Calculus II
Practice Exam 2, Answers

Find the indefinite integrals of the following functions.

1. $\int \frac{\arccos x}{\sqrt{1-x^2}} dx$

Answer. Let $u = \arccos x,\ du = -dx/\sqrt{1-x^2}$. Then

$$\int \frac{\arccos x}{\sqrt{1-x^2}} dx = -\int u du = -\frac{u^2}{2} + C = -\frac{(\arccos x)^2}{2} + C\ .$$

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2. $\int \frac{(\ln x + 1)^2}{x} dx$

Answer. Let $u = \ln x + 1,\ du = dx/x$. We get

$$\int \frac{(\ln x+1)^2}{x} dx = \int u^2 du = \frac{(\ln x+1)^3}{3} + C\ .$$

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3. $\int \cos^3 x \sin^2 x dx$

Answer. Here we use the famous trig identity: $\cos^2 x = 1 - \sin^2 x$:

$$\int \cos^3 x \sin^2 x dx = \int (1-\sin^2 x)\sin^2 x \cos x dx\ .$$

Now, let $u = \sin x,\ du = \cos x dx$:

$$= \int (1-u^2)u^2 du = \int (u^2 - u^4) du = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C\ .$$

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4. $\int \frac{dx}{x^2(x-1)}$

Answer. We need the partial fraction expansion

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}\ .$$

Putting the right hand side over a common denominator, we get this equality of the numerators: $1 = Ax(x-1) + B(x-1) + Cx^2$.

$$x = 0:\ 1 = -B \quad \text{so that} \quad B = -1$$

$$x = 1:\ 1 = C$$

$$\text{coefficient of } x^2:\ 0 = A + C\ , \quad \text{so that} \quad A = -1\ .$$

Thus

$$\int \frac{dx}{x^2(x-1)} = -\int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{dx}{x-1} = -\ln x + \frac{1}{x} + \ln(x^2-1) + C\ .$$

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5. $\int \sqrt{x}(x+1)dx = \int (x^{3/2}+x^{1/2})dx = \frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+C$.

This is here to remind you to try the easy things first; don't just assume that if an integral appears in this section, it will need some complicated substitution or integration by parts.

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6. $\int \frac{dx}{x(x^2+4x+5)} =$

Answer. Again, we seek the partial fractions decomposition

$$\frac{1}{x(x^2+4x+5)} = \frac{A}{x} + \frac{Bx+C}{(x^2+4x+5)} .$$

This leads to the equation of numerators: $1 = A(x^2+4x+5)+Bx^2+Cx$.

$$x = 0: \ 1 = 5A \quad \text{so that} \quad A = \frac{1}{5}$$

$$\text{coefficient of } x: \ 0 = 4A+C , \quad \text{so that} \quad C = -\frac{4}{5}$$

$$\text{coefficient of } x^2: \ 0 = A+B , \quad \text{so that} \quad B = -\frac{1}{5} .$$

This gets us to

$$\int \frac{dx}{x(x^2+4x+5)}dx = \frac{1}{5}(\int \frac{dx}{x} - \int \frac{x+4}{x^2+4x+5}dx) .$$

To integrate the last term we use a little algebra:

$$\int \frac{x+4}{x^2+4x+5}dx = \int \frac{(x+2)dx}{(x+2)^2+1} + \int \frac{2dx}{(x+2)^2+1}$$

$$= \frac{1}{2}\ln((x+2)^2+1)+2\arctan(x+2)+C .$$

So, finally

$$\int \frac{dx}{x(x^2+4x+5)} = \frac{1}{5}(\ln x - (\frac{1}{2}\ln((x+2)^2+1)+2\arctan(x+2))+C .$$

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7. $\int x^2 \sin x dx$

Answer. Let $u = x^2, \ dv = \sin x dx, \ du = 2xdx, v = -\cos x$:

$$\int x^2 \sin x dx = -x^2\cos x + 2\int x\cos x dx .$$

Another integration by parts handles the last integral: $u = x, \ dv = \cos x dx, du = dx, \ v = \sin x$:

$$\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x ,$$

finally giving

$$\int x^2\sin x dx = -x^2\cos x + 2(x\sin x+\cos x)+C .$$

Calculate the definite integral.

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8. $\int_0^2 (x^2+3x-1)^2(2x+3)dx$

Answer. Here the substitution $u = x^2+3x-1,\ du = 2x+3$ works:

$$\int_0^2 (x^2+3x-1)^2(2x+3)dx = \int_{-1}^9 u^2 du = \frac{u^3}{3}\Big|_{-1}^9 = 3^5 + \frac{1}{3} \ .$$

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9. $\int_1^e x^2 \ln(2x)dx$

Answer. Integrate by parts with $u = \ln(2x),\ dv = x^2dx,\ du = dx/x,\ v = x^3/3$:

$$\int x^2 \ln(2x)dx = \frac{x^3}{3}\ln(2x) - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln(2x) - \frac{x^3}{9} \ .$$

Then

$$\int_1^e x^2 \ln(2x)dx = (\frac{x^3}{3}\ln(2x) - \frac{x^3}{9})\Big|_1^e = \frac{e^3}{3}\ln(2e) - \frac{e^3}{9} - \frac{1}{3}\ln(2) - \frac{1}{9} \ .$$

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10. $\int_0^2 \frac{dx}{x^2+4x+5} = \int_0^2 \frac{dx}{1+(x+2)^2} = \arctan(x+2)\Big|_0^2 = \arctan 4 - \arctan 2 = .21867$.