

Calculus II
Practice Exam 1, Answers

1. Differentiate:

a) $f(x) = \ln\left(\frac{x+1}{x-1}\right)$

Answer. Using the laws of logs, we can write $f(x) = \ln(x+1) - \ln(x-1)$. Now the problem is easier:

$$f'(x) = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{x^2-1}.$$

b) $g(x) = 5^x \log_5 x$

Answer. Write $5^x = e^{(\ln 5)x}$, and $\log_5 x = \ln x / \ln 5$, so that

$$g(x) = (e^{(\ln 5)x}) \frac{\ln x}{\ln 5}.$$

Now, by the product rule

$$g'(x) = (\ln 5)(e^{(\ln 5)x}) \frac{\ln x}{\ln 5} + (e^{(\ln 5)x}) \frac{1}{x \ln 5} = 5^x \ln x + \frac{5^x}{x \ln 5}.$$

c) $h(x) = 5^{\log_2(x^2+1)}$

Answer. Write $\log_2(x^2+1) = \ln(x^2+1) / \ln 2$ and $5 = e^{\ln 5}$ so that

$$5^{\log_2(x^2+1)} = e^{\ln(x^2+1)(\ln 5 / \ln 2)} = (x^2+1)^{\ln 5 / \ln 2}.$$

Now, we can differentiate:

$$h'(x) = \frac{\ln 5}{\ln 2} (x^2+1)^{(\ln 5 / \ln 2) - 1} (2x).$$

2. Differentiate:

a) $f(x) = \frac{e^x}{x^2}$

Answer. By the quotient rule: $f'(x) = \frac{x^2 e^x - 2x e^x}{x^4} = \frac{x e^x - 2}{x^3}.$

b) $g(x) = 5^{x^2+1} = e^{(\ln 5)(x^2+1)}$

Answer. $g'(x) = e^{(\ln 5)(x^2+1)} (\ln 5) (2x) = 2x 5^{x^2+1}.$

Note: The above problems have been selected so as to exhibit as much as possible of the rules of exponents and logarithms as possible. The problems on the exam will be less complex.

3. Integrate:

a) $\int e^{\tan x} \sec^2 x dx$

Answer. Let $u = \tan x$, $dy = \sec^2 x dx$. Then

$$\int e^{\tan x} \sec^2 x dx = \int e^u du = e^u + C = e^{\tan x} + C .$$

b) $\int_0^3 2^{x^2} x dx$

Answer. Let $u = x^2$, $du = 2x dx$. At $x = 0, 3$, we have $u = 0, 9$. Thus

$$\int_0^3 2^{x^2} x dx = \frac{1}{2} \int_0^9 e^{(\ln 2)u} du = \frac{e^{(\ln 2)u}}{2 \ln 2} \Big|_0^9 = \frac{2^9 - 1}{2 \ln 2} .$$

Notice that I always change A^B to $e^{(\ln A)B}$, simply because I can never remember, in integrating A^u if I have to divide by, or multiply by, $\ln A$. This way I just remember the one rule $\int e^{ax} dx = e^{ax}/a + C$.

4. Solve the initial value problem: $y' = x(5 - y)$, $y(0) = 1$.

Answer. The variables separate to give:

$$\frac{dy}{5 - y} = x dx .$$

Integrating both sides, we have

$$-\ln |5 - y| = \frac{x^2}{2} + C ,$$

and exponentiating gives $|5 - y| = Ke^{-x^2/2}$. The initial condition leads to the equation $|5 - 1| = K$, so $K = 4$. The ambiguity in the absolute value disappears since $5 - 1$ is positive, so the solution is $y = 5 - 4e^{-x^2/2}$.

Note: If the initial condition were $y(0) = 9$, then we'd still have $K = 4$, but since $5 - 9$ is negative, we have to evaluate $|5 - y|$ as $y - 5$, and the solution we get is $y = 5 + 4e^{-x^2/2}$.

5. If I invest \$4000 in a fund, with an interest rate of 8%, compounded continuously, how long will it take for the fund to be worth \$10,000?

Answer. Putting the given data in the basic growth equation gives $10000 = 4000e^{(.08)t}$. Solve for t : $t = (\ln(10/4))/(.08) = 11.45$ years.

6. I invested \$6000 3 years ago in a fund bearing continuously compounded interest. Now that fund is worth \$8000. Assuming the same rate of interest, how much will an investment of \$10,000 be worth in 5 years?

Answer. Let r be the interest rate, our answer, P , is given by $P = 10e^{5r}$. We find r from the given data: $8 = 6e^{3r}$. From this, we find $r = .0959$. Now solve for P : $P = 10e^{(.0959)5} = 16,153$ dollars.

7. Carbon¹⁴ has a half life of 5801 years. How long does it take for a sample to be reduced to 80% its original size?

Answer. From the decay equation $P = P_0 e^{rt}$, we have the two equations:

$$.5 = e^{5801r} , \quad .8 = e^{rt} ,$$

where t is the solution to our problem. The first equation gives $r = \frac{\ln(.5)}{5801}$, so, from the second equation

$$t = \frac{1}{r} \ln(.8) = \frac{\ln(.8)}{\ln(.5)} 5801 = 1868 \text{ years} .$$

8. A certain autocatalytic chemical reaction proceeds according to the differential equation

$$\frac{dx}{dt} = .01 \frac{1-x^2}{x}$$

where $x(t)$ is the fraction of the sample which is the resulting compound at time t , in seconds. If we start with $x(0) = .2$, how long does it take for x to reach .9?

Answer. The variables separate in the equation, giving

$$\frac{xdx}{1-x^2} = .01dt ,$$

which integrates to

$$-\frac{1}{2} \ln(1-x^2) = .01t + C , \quad \text{or} \quad \ln(1-x^2) = -.02t + C .$$

This exponentiates to $1-x^2 = Ke^{-.02t}$. Now at $t = 0$, $x = .2$, giving $K = 1 - .04 = .96$. The solution is then given by the equation

$$x^2 = 1 - .96e^{-.02t} .$$

We want to solve for t when $x = .9$ (so $x^2 = .81$). This gives

$$e^{-.02t} = 1 - .81 = .19 \quad \text{or} \quad t = -50 \ln \frac{19}{96} = 81 \text{ seconds} .$$

9. Solve the initial value problem $xy' - y = x^3$, $y(1) = 2$.

Answer. First solve the homogeneous equation $xy' - y = 0$, for which the variables separate: $dy/y = dx/x$. This has the solution $y = Kx$. So, we try $y = ux$ in the original equation, getting

$$x^2u' = x^3 \quad \text{or} \quad u' = x ,$$

which has the solution $u = x^2/2 + C$. Thus

$$y = ux = \frac{x^3}{2} + Cx .$$

The initial condition gives $2 = 1/2 + C$, so $C = 3/2$, and the answer is

$$y = \frac{x^2 + 3x}{2} .$$

10. Solve the initial value problem $y' - 2xy = e^{x^2}$, $y(0) = 4$.

Answer. First solve the homogeneous equation $y' - 2xy = 0$, for which the variables separate: $dy/y = 2xdx$. This has the solution $y = Ke^{x^2/2}$. So try $y = ue^{x^2/2}$ in the original equation, getting

$$e^{x^2/2}u' = e^{x^2/2} , \quad \text{or} \quad u' = 1 .$$

Thus $u = x + C$, and

$$y = ue^{x^2/2} = (x + C)e^{x^2/2} .$$

From the initial conditions we get $C = 4$, so $y = (x + 4)e^{x^2/2}$.