Mathematics 1220-90 Summer, 2003, Final Examination Answers

1. Find the integral : $\displaystyle\int \frac{\ln x}{x^2}dx$

Answer. We integrate by parts to replace the term $\ln x$ by a monomial. Make the substitution $u = \ln x,\ dv = dx/x^2,\ du = dx/x,\ v = -1/x$. Then

$$\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C \ .$$

2. Integrate $\displaystyle\int_1^4 (x^2 + 3x)\sqrt{x}dx$

Answer. Just do the multiplication and integrate;

$$\int_1^4 (x^{5/2} + 3x^{3/2})dx = \frac{2}{7}x^{7/2} + 3\frac{2}{5}x^{5/2}|_1^4 = \frac{254}{7} + 3\frac{62}{5} = 73.486 \ .$$

3. Integrate : $\displaystyle\int \frac{u^2+1}{u^2(u-1)}du$

Answer. We integrate by parts; that is, we find $A,\ B,\ C$ such that

$$\frac{u^2+1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1} \ .$$

Putting the right side over a common denominator and equating numerators gives $u^2+1 = Au(u-1) + B(u-1) + Cu^2$. Now evaluate at the roots:

$$\text{At } u = 0 \ : \qquad 1 = B(-1) \quad \text{so that} \quad B = -1 \ ,$$

$$\text{At } u = 1 \ : \qquad 1^2 + 1 = C \quad \text{so that} \quad C = 2 \ .$$

Now we equate the coefficients of u^2 : $1 = A + C$, so that $A = -C + 1 = -1$. This gives us

$$\int \frac{u^2+1}{u^2(u-1)}du = \int \frac{-1}{u}du + \int \frac{-1}{u^2}du + \int \frac{2}{u-1}du = -\ln|u| + u^{-1} + 2\ln|u-1| + C \ .$$

4. Four years ago I invested \$10,000 in an account bearing continuously compounded interest. Today I have \$13,500. Assuming that the same interest rate continues into the future, when will my account have \$20,000?

Answer. We use the equation $P = P_0 e^{rt}$. We are given: at $t = 0$, $P_0 = 10000$; at $t = 4$, $P = 13500$. Thus we can solve for the interest rate;

$$135 = 100e^{4r} \ , \quad \text{giving} \quad r = \frac{\ln(1.35)}{4} = .075 \ .$$

Now we want to know how long it takes \$13500 to grow to \$20000 at that rate:

$$200 = 135e^{.075t} \quad \text{giving} \quad t = \frac{\ln(200/135)}{.075} = 5.241$$

more years.

5. Find the limit. Show your work.

a) **Answer**.
$$\lim_{x\to 4} \frac{\sin(\pi x)}{x^2 - 16} =^{l'H} \lim_{x\to 4} \frac{\pi\cos(\pi x)}{2x} = \frac{\pi}{8} \ .$$

b) **Answer**.
$$\lim_{x\to 0} \frac{e^x - 1 - x}{2x^2} =^{l'H} \lim_{x\to 0} \frac{e^x - 1}{4x} =^{l'H} \lim_{x\to 0} \frac{e^x}{4} = \frac{1}{4} \ .$$

c) **Answer**.
$$\lim_{x\to\infty} \frac{x^3}{e^x} =^{l'H} \lim_{x\to\infty} \frac{3x^2}{e^x} =^{l'H} \lim_{x\to\infty} \frac{6x}{e^x} =^{l'H} \lim_{x\to\infty} \frac{6}{e^x} = 0 \ .$$

6. Do the integrals converge? If so, evaluate:

a) **Answer**.
$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{A\to\infty} \int_1^A \frac{dx}{1+x^2} = \lim_{A\to\infty} (\arctan A - \arctan 1) = \frac{\pi}{4} \ .$$

b) **Answer**.
$$\int_1^\infty \frac{dx}{1+x} = \lim_{A\to\infty} \int_1^A \frac{dx}{1+x} = \lim_{A\to\infty} (\ln(1+A) - \ln 2) = \infty \ .$$

7. Find the sum of the series. If the series does not converge, just write "DIV". Carefully note the limits of summation.

a)
$$\sum_{n=0}^\infty \frac{2^{n-1}}{3^n}$$

Answer. This is nearly the geometric series. Thus, we write

$$\sum_{n=0}^\infty \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum_{n=0}^\infty (\frac{2}{3})^n = \frac{1}{2} \frac{1}{1 - 2/3} = \frac{3}{2} \ .$$

b) $$\sum_{n=3}^{\infty} \frac{1}{n(n-1)}$$

Answer. This is a telescoping series:

$$\sum_{n=3}^{\infty} \frac{1}{n(n-1)} = \sum_{n=3}^{\infty} (\frac{1}{n-1} - \frac{1}{n}) = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = \frac{1}{2} .$$

c) $$\sum_{n=0}^{\infty} \frac{n}{3^{n-1}}$$

Answer. Thius series suggests the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n .$$

Now, the factor n suggests differentiating:

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} .$$

Now, substitute $x = 1/3$:

$$\sum_{n=0}^{\infty} \frac{n}{3^{n-1}} = \frac{1}{(1-\frac{1}{3})^2} = \frac{9}{4} .$$

d) $$\sum_{n=0}^{\infty} \frac{(12)^n}{n!}$$

Answer. This is just the series for e^x at $x = 12$, so the sum is e^{12}.

8. Consider the hyperbola given by the equation $x^2 - 2y^2 - 2x + 12y = 138$.
a) What is its center? b) What is the distance between the its vertices?

Answer. Complete the square: $x^2 - 2x + 1 - 2(y^2 - 6y + 9) = 138 + 1 - 18$, which becomes

$$(x-1)^2 - 2(y-3)^2 = 121 \quad \text{or} \quad \frac{(x-1)^2}{121} - \frac{(y-3)^2}{121/2} = 1 .$$

Thus the center of the hyperbola is at (1,3), and its major axis is the line $y = 3$. Setting $y = 3$, we find $x - 1 = \pm 11$, so the vertices are at (-10,3),(12,3). Thus the distance between the vertices is 22.

9. Find the area of the region enclosed by the curve given in polar coordinates by $r = 2\cos\theta$.

Answer. This is the circle of radius 1 centered at the point (1,0) so has area π. If you did not recognize the curve, you integrated, but only from 0 to π, since that is all you need to traverse the whole boundary. Thus

$$Area = \int_0^\pi \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_0^\pi (2\cos\theta)^2 d\theta = \int_0^\pi (1+\cos(2\theta))d\theta = \pi \ .$$

10. a) Find the general solution of the differential equation $y'' - 6y' + 5y = 0$.

Answer. The auxiliary equation, $r^2 - 6r + 5 = 0$ has the roots $r = 1, 5$. Thus the general solution is

$$y_h = Ae^x + Be^{5x} \ .$$

b) Solve the initial value problem:

$$y'' - 6y' + 5y = 10 \quad , \qquad y(0) = 0, y'(0) = 0 \ .$$

Answer. A particular solution is the constant function $y_p = 2$. Thus the general solution is $y = y_p + y_h = 2 + Ae^x + Be^{5x}$. We solve for A and B from the initial conditions:

$$0 = 2 + A + B \ , \quad 0 = A + 5B \ , \quad \text{so} \quad A = -\frac{5}{2}, \ B = \frac{1}{2}$$

and the solution is

$$y = 2 - \frac{5}{2}e^x + \frac{1}{2}e^{5x} \ .$$

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