**Mathematics 1220-90 Final Examination. Answers**

1. Find the integrals:

*a)* $$\int_2^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

**Solution**. Let $u = x^{1/2},\ du = (1/2)x^{-1/2}dx$. The integral becomes

$$\int_2^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2\int_{\sqrt{2}}^2 e^u du = 2(e^2 - e^{\sqrt{2}}) \ .$$

*b)* $$\int_0^2 \frac{x^2}{1+x^2} dx$$

**Solution**. First do some algebra:

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2} \ .$$

Thus

$$\int_0^2 \frac{x^2}{1+x^2} dx = \int_0^2 (1 - \frac{1}{1+x^2}) = x - \arctan x\Big|_0^2 = 2 - \arctan 2 \ .$$

2. Integrate $$\int \frac{u+1}{u(u-1)} du$$

**Solution**. First, we find the partial fractions expansion:

$$\frac{u+1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1)+Bu}{u(u-1)} \ .$$

Equate numerators: $u+1 = (A+B)u - A$, so $A+B = 1,\ -A = 1$ from which we get $A = -1,\ B = 2$. Thus

$$\int \frac{u+1}{u(u-1)} du = -\int \frac{du}{u} + 2\int \frac{du}{u-1} = -\ln u + 2\ln(u-1) + C \ .$$

3. Integrate $$\int xe^x dx$$

**Solution**. We integrate by parts with $u = x,\ du = dx,\ dv = e^x dx,\ v = e^x$.

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C \ .$$

4. The population of Sourwater Canyon, New Mexico has been continuously decreasing at a steady rate for decades. Assuming continued decay at the same rate, if the population ten years ago was 8,000 and today it is 5,000, when will there be only 2 people left in Sourwater Canyon?

**Solution**. We use the basic growth equation: $P = P_0 e^{rt}$. To find $r$, put in the given data: $P = 8000, P_0 = 5000, t = -10$:

$$8000 = 5000e^{-10r} \quad \text{so that} \quad r = -\frac{1}{10}\ln(\frac{8}{5}) = -.0470 \ .$$

Now, to find $t$ for $P = 2$, we start with

$$2 = 5000e^{-.0470t} \quad \text{so} \quad r = -\frac{\ln(2/5000)}{.0470} = \frac{\ln(2500)}{.0470} = 166.47 \text{ years} \ .$$

5. Find the limit. Show your work.

*a*)

$$\lim_{x\to 2}\frac{x-2}{x^2-4} =^{l'H} \lim_{x\to 2}\frac{1}{2x} = \frac{1}{4}$$

Alternatively, do the algebra:

$$\frac{x-2}{x^2-4} = \frac{1}{x+2} \to \frac{1}{4}$$

*b*)

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{1/x} =^{l'H} \lim_{x\to 0}\frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0$$

*c*)

$$\lim_{x\to\infty}\frac{x^2}{(2x+1)^2} =^{l'H} \lim_{x\to\infty}\frac{2x}{2(2x+1)(2)} =^{l'H} \frac{2}{2\cdot 2\cdot 2} = \frac{1}{4} \ .$$

6. Find the integral

*a*)

$$\int_2^\infty \frac{dx}{x^{\frac{10}{9}}}$$

$$\int_2^A x^{\frac{-10}{9}}\,dx = -9x^{\frac{-1}{9}}\Big|_2^A = 9(2^{\frac{-1}{9}} - A^{\frac{-1}{9}}) \to 9(2^{\frac{-1}{9}})$$

as $A \to \infty$.

*b*)

$$\int_0^2 \frac{dx}{x^{\frac{9}{10}}}$$

$$\int_{\epsilon}^{2} x^{\frac{-9}{10}}\,dx = 10x^{\frac{1}{10}}\Big|_{\epsilon}^{2} = 10(2^{\frac{1}{10}} - \epsilon^{\frac{1}{10}}) \to 10(2^{\frac{1}{10}})$$

as $\epsilon \to 0$.

7. The function $f(x)$ is defined for $-3 \le x \le 3$, and has the Maclaurin series at the origin:

$$f(x) = \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} x^n \ .$$

a) What is the radius of convergence of this series?

**Solution**. We use the ratio test:

$$\frac{(n+2)^2}{(n+1)!}\frac{n!}{(n+1)^2} = \Big(\frac{1+2/n}{1+1/n}\Big)^2 \frac{1}{n+1} \to 0 \ ,$$

so, since the radius of convergence is the inverse of this limit, $R = \infty$.

b) What is the Maclaurin series for $F(x) = \int_0^x f(t)dt$?

**Solution**. We integrate the series term by term:

$$F(x) = \int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!}\frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{n+1}{n!} x^{n+1} \ .$$

c) What is the Maclaurin series for $x^2F(x)$?

**Solution**. We multiply the series by $x^2$:

$$x^2F(x) = \sum_{n=0}^{\infty} \frac{n+1}{n!} x^{n+3} \ .$$

8. Find the focus of the parabola given by the equation $x^2 - 8y + 2x + 17 = 0$.

**Solution**. First we complete the square: $x^2 + 2x + 1 - 8y + 16 = 0$, or $8(y-2) = (x+1)^2$. The vertex of this parabola is (-1,2), the axis is the line $x = -1$, and the parabola opens upwards. Comparing with the standard form $x^2 = 4py$, we see that $4p = 8$, so the distance from vertex to focus is $P = 2$. The focus is 2 units above the vertex, so is at (-1,4).

9. Find the area of the region enclosed by the curve given in polar coordinates by $r = 2\cos\theta\sqrt{\sin\theta}$, $0 \le \theta \le \pi/2$.

**Solution**. .

$$Area = \int \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2}(4\cos^2\theta \sin\theta) d\theta = -\frac{2}{3}\cos^3\theta d\theta \Big|_0^{\pi/2} = \frac{2}{3} \ .$$

10. Find the general solution, for $x > 0$ of the differential equation

$$x\frac{dy}{dx} + \ln x = 0 \ .$$

**Solution**. Here we can separate the variables to get:

$$dy = -\frac{\ln x}{x} dx = 0 \ .$$

Taking integrals of both sides, we get

$$y = -\frac{1}{2}(\ln x)^2 + C \ .$$