Calculus II
Exam 4, Spring 2003, Answers

1. Find the focus and vertex (or foci and vertices) of the conic given by the equation $x^2 - 8x - 8y = 8$.

Answer. Complete the square:

$$x^2 - 8x + 16 - 8y = 8 + 16 , \quad \text{or} \quad (x-4)^2 = 8(y+3) .$$

This is the equation of a parabola which opens upward, and whose vertex is at (4,-3). Since $4p = 8$, $p = 2$, so the focus is 2 units above the vertex, at (4,-1).

---

2. Find the equation of the conic which has a focus at (6,2) and ends of the minor axis at (1,7) and (1,-3).

Answer. The center of the conic is midway between the ends of the minor axis, so is at $C : (1, 2)$. Thus the axes are the lines $x = 1$, $y = 2$, and $b = 5$. Since a focus is at (6,2), $c = 6 - 1 = 5$. Thus $a^2 = b^2 + c^2 = 25 + 25$, so $a = 5\sqrt{2}$. Since the center is at (1,2), and $a^2 = 50$, $b^2 = 25$, we have the equation

$$\frac{(x-1)^2}{50} + \frac{(y-2)^2}{25} = 1 .$$

If you first thought the conic might be a hyperbola, and tried $c^2 = a^2 + b^2$ first, you would obtain $a = 0$, so that excludes that possibility.

---

3. Find the equation of the tangent line ot the hyperbola

$$\frac{x^2}{4} - y^2 = 1$$

at the point $(4, \sqrt{3})$.

Answer. Taking differentials, we obtain

$$\frac{xdx}{2} - 2ydy = 0 .$$

Putting in the values $x = 4$, $y = \sqrt{3}$ gives us $2dx - 2\sqrt{3}dy = 0$, so the tangent line has slope $dy/dx = 1/\sqrt{3}$. The equation thus is

$$y - \sqrt{3} = \frac{x-4}{\sqrt{3}} .$$

---

4. Find the area of the region that lies outside the circle $r = 1$ and inside the circle $r = 2\cos\theta$.

Answer. At the point of intersection of the two curves we have $1 = 2\cos\theta$, so $\theta = \pm\pi/3$. Since the area inside a curve $r = r(\theta)$ is given by integrating $dA = (1/2)r^2 d\theta$, the area between the curves $r = 2\cos\theta$ and $r = 1$ is

$$Area = \frac{1}{2}\int_{-\pi/3}^{\pi/3}[(2\cos\theta)^2 - 1^2]d\theta = \int_{-\pi/3}^{\pi/3}(1 + 2\cos(2\theta) - \frac{1}{2})d\theta = \sqrt{3}$$

---

5. Find the center, foci and vertices of the ellipse given in polar coordinates by the equation

$$r = \frac{6}{1 + \frac{1}{2}\sin\theta} \ .$$

Answer. This ellipse has a focus at the origin, and its vertices are at the points where r has a minimum and a maximum. The minimum is attained when $\sin\theta$ is as large as it can be, so is at $\theta = \pi/2$, with $r = 12$. The maximum is at $\theta = -\pi/2$. with $r = 4$. Thus the y-axis is the major axis of the ellipse, and the vertices are at (0,12), (0,-4). The center is the midpoint of this segment, so is at (0,4). Thus $c = 4$ (the distance from the center to a focus, and $a = 8$, the distance from the center to a vertex. In summary:

$$\text{Center}: (0,4) \quad \text{Foci}: (0,0),\ (0,8) \quad \text{Vertices}: (0,-4),\ (0,12)\ .$$

For the record, since $b^2 = a^2 - c^2 = 64 - 16 = 48,\ b = 4\sqrt{3}$, and the equation of the ellipse in cartesian coordinates is

$$\frac{x^2}{48} + \frac{(y-4)^2}{64} = 1\ . \tag{1}$$

An alternative method is to switch to cartesian coordinates first. We can rewrite the equation as $r(1 + \sin\theta/2) = 6$ or, in cartesian coordinates, $\sqrt{x^2+y^2} + y/2 = 6$. Moving the term $y/2$ to the right hand side and squaring we get

$$x^2 + y^2 = 36 - 6y + \frac{y^2}{4}$$

which brings us to (1) after combining terms and completing the square.