Mathematics 1220 Calculus II, Examination 4, Answers

1. Consider the conic given by the equation $4x^2 - 24x + 9y^2 + 18y + 9 = 0$.
a) What kind of conic is it?

Solution. Complete the square to get

$$4(x^2 - 6x + 9) + 9(y^2 + 2y + 1) = -9 + 36 + 9 = 36 ,$$

so the conic is an ellipse.

b) Give the coordinates of its vertex/vertices.

Solution. Now we put the equation in normal form:

$$\frac{(x-3)^2}{9} + \frac{(y+1)^2}{4} = 1 .$$

The center of the ellipse is at (3,-1), and the major radius is 3, and the minor radius is 2. Thus the major axis is the line $y = -1$, and the vertices are each 3 units from the center, so are at (6,-1) and (0,-1).

c) Give the coordinates of its focus/foci.

Solution. The distance of the foci from the center is c where $c^2 = a^2 - b^2 = 9 - 4 = 5$. Thus the foci are at $(3 \pm \sqrt{5}, -1)$.

2. Consider the conic given by the equation $4x^2 - 24x + 9y + 18 = 0$.
a) What kind of conic is it? b) and c): where is the vertex and focus?

Solution. Complete the square to get

$$4(x^2 - 6x + 9) + 9y - 18 = 0 \quad \text{or} \quad y - 2 = -\frac{4}{9}(x-3)^2 ,$$

so the conic is a parabola with vertex at (3,2) and $4p = -4/9$. The parabola opens downward, and is p units below the vertex, so is at (3-(1/9),2).

3. Find the equation of the hyperbola whose vertices are at (-3,0), (3,0) and which goes through the point (6,6).

Solution. The center is at the origin, and the major axis is the x-axis, so the equation of the hyperbola is of the form

$$\frac{x^2}{9} - \frac{y^2}{b^2} = 1 .$$

We solve for b using the fact that (6,6) is on the curve:

$$\frac{36}{9} - \frac{36}{b^2} = 1 \quad \text{so} \quad \frac{36}{b^2} = 3 \ ,$$

and thus $b^2 = 12$. This gives the equation

$$\frac{x^2}{9} - \frac{y^2}{12} = 1 \ .$$

4. Find the length of the spiral $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

Solution. We start with the equation $ds^2 = dr^2 + r^2 d\theta^2$. Now, $dr = 2e^{2\theta}d\theta$, so

$$ds^2 = 4e^{4\theta}d\theta^2 + e^{4\theta}d\theta^2 = 5e^{4\theta}d\theta^2 \ .$$

Thus $ds = \sqrt{5}e^{2\theta}d\theta$, and the length is

$$\int_0^{2\pi} \sqrt{5}e^{2\theta}d\theta = \frac{\sqrt{5}}{2}e^{2\theta}\Big|_0^{2\pi} = \frac{\sqrt{5}}{2}(e^{4\pi} - 1) \ .$$

5. Find the area enclosed by the limaçon $r = 3 + 2\sin\theta$.

Solution. $dA = (1/2)r^2 d\theta = (1/2)(3 + 2\sin\theta)^2 d\theta = (1/2)((+12\sin\theta + 4\sin^2\theta)d\theta$. Thus the area is (and, in order to integrate, we have used the half-angle formula on the last term:

$$\frac{1}{2}\int_0^{2\pi} (9 + 12\sin\theta + 2(1 - \cos 2\theta))d\theta = 11\pi \ ,$$

since the integrals of the trigonometric terms is zero.