Calculus II
Exam 4, Fall 2002, Answers

1. Find the foci of the ellipse given by the equation $x^2+4y^2+2x=8$.

Answer. Complete the square:

$$(x^2+2x+1)+4y^2=8+1 \qquad \text{so that} \qquad (x+1)^2+4y^2=9$$

giving us

$$\frac{(x+1)^2}{9}+\frac{y^2}{9/4}=1$$

so the center is at (-1,0), the axis is horizontal, so is the line $y=0$, and $a^2=9$, $b^2=9/4$, so $c^2=9-9/4=27/4$. Thus the foci are $\sqrt{27}/2$ units removed from the center along the line $y=0$, so are at $(-1\pm\sqrt{27}/2,0)$.

---

2. The point $P(1,5)$ lies on the parabola given by the equation $y^2-8x-2y=7$. Let F be the focus of this parabola.

a) What are the coordinates of the focus F?

Answer. Complete the square;

$$y^2-2y+1=8x+7+1 \qquad \text{so that} \qquad (y-1)^2=8(x+1)\ .$$

Thus the vertex is at (-1,1), the axis is horizontal, and the parabola opens to the right. Since $4p=8$, the focus of the parabola is two units to the right of the vertex on the axis, so is at (1,1).

b) What is the angle between the line PF and the tangent to the parabola at P?

Answer. By the focal property of the parabola, this is the same as the angle between the tangent at P and the horizontal. We find the slope of that line by differentiating the equation of the parabola and evaluating at $P(1,5)$:

$$2(y-1)\frac{dy}{dx}=8 \qquad \text{so that} \qquad 2(5-1)\frac{dy}{dx}=8$$

or $dy/dx=1$, and the angle has tangent 1, so is $\pi/4$.

Another way to see this is to note that the point P lies on the same vertical as the focus F, so PF makes an angle of $\pi/2$ with the horizontal. Thus, if α is the angle between the line PF and the tangent at P, we have $\alpha=\pi/2+\alpha=\pi$, so $\alpha=-\pi/4$.

---

3. Find the equation of the ellipse with vertices at $(0,\pm2)$ and foci at $(0,\pm1)$.

Answer. For this ellipse, the axis is the line $x=0$ and the center is the origin (midway between the vertices). We have $c=1$, $b=2$, and since $c^2=b^2-a^2=4-1$, we have $a=\sqrt{3}$. Thus the equation is

$$\frac{x^2}{3}+\frac{y^2}{4}=1$$

---

4. Find the integral (do not try to evaluate it) giving the length of the spiral $r=2\theta$ from $\theta=0$ to $\theta=2\pi$.

Answer. Since $dr = 2d\theta$, $ds^2 = dr^2 + r^2 d\theta^2 = (4+4\theta^2)d\theta^2$, so the length is

$$Length\ = \int_0^{2\pi} 2\sqrt{1+\theta^2}d\theta\ .$$

---

5. Find the area enclosed by the cardiod $r = 2 + 2\sin\theta$.

Answer. $dA = (1/2)r^2 d\theta = (1/2)(2+2\sin\theta)^2 d\theta = 2(1+\sin\theta)^2 d\theta$ and

$$Area\ = 2\int_0^{2\pi}(1+\sin\theta)^2 d\theta = 2\int_0^{2\pi}(1+2\sin\theta+\frac{1-\cos 2\theta}{2})d\theta = 6\pi$$