Calculus II
Exam 3, Spring 2003, Answers

Remember : you MUST show your work.

1. Find the limits

a) $\lim_{x\to\pi/2^+}(\tan x)(x-\pi/2)$

Answer. First of all, use the identity $\tan x = \sin x \cos x$, and the fact that the limit of a product is the product of the limits to obtain

$$= \lim_{x\to\pi/2^+} \sin x \lim_{x\to\pi/2^+} \frac{x-\pi/2}{\cos x} =^{l'H} \sin(\pi/2) \lim_{x\to\pi/2^+} \frac{1}{-\sin x} = -1 \, .$$

b) $\lim_{x\to\infty} \frac{e^{x+2}}{e^{2x}}$

Answer. Using the laws of exponents: $e^{x+2}/e^{2x} = e^2e^{-x}$, so

$$= \lim_{x\to\infty} \frac{e^2}{e^x} = 0 \, .$$

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2. Does the integral converge or diverge? Give reasons. If you can, evaluate the integral.

a) $\int_3^\infty \frac{dx}{x(\ln x)^2}$ Converges.

Answer. Let $u = \ln x,\ du = dx/x$. Then integrate from 3 to A:

$$\int_3^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 3}^{\ln A} u^{-2}du = -u^{-1}\Big|_{\ln 3}^{\ln A} = \frac{1}{\ln 3} - \frac{1}{\ln A} \to \frac{1}{\ln 3}$$

as $A \to \infty$.

b) $\int_0^1 \frac{dx}{(x-1)^2}$ Diverges.

Answer. We calculate the integral from 0 to c for c slightly less than 1:

$$\int_0^c \frac{dx}{(x-1)^2} = -(x-1)^{-1}\Big|_0^c = \frac{1}{1-c} - 1 \to \infty$$

as $c \to 1$.

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3. Does the series converge or diverge? Give reasons.

a) $\sum_{n=0}^{\infty} \frac{e^{-n}}{n^e}$

Answer. This series converges. Use comparison with the geometric series:

$$\frac{e^{-n}}{n^e} < e^{-n} = (\frac{1}{e})^n \, ,$$

and $1/e < 1$.

b) $\sum_{n=0}^{\infty} \frac{3n^2 - 5n + 17}{4n^3 + 25n + 1}$

Answer. This series diverges by comparison with a p-series, $p < 1$:

$$\frac{3n^2 - 5n + 17}{4n^3 + 25n + 1} = \frac{1}{n} \frac{3 - 5/n + 17/n^2}{4 + 25/n + 1/n^2} \geq \frac{1}{n}$$

eventually, since the second fraction converges to 3/4.

c) $\sum_{n=0}^{\infty} \frac{5n}{(n^2+1)^2}$

Answer. This series converges by comparison:

$$\frac{5n}{(n^2+1)^2} < \frac{5n}{(n^2)^2} = \frac{5}{n^3} \; .$$

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4. What is the interval of convergence of the power series? Show your work.

a) $\sum_{n=0}^{\infty} 5^n (x-2)^n$

Answer. If we write the series as

$$\sum_{n=0}^{\infty} [5(x-2)]^n \; ,$$

by comparison with the geometric series, this converges if $|5(x-2)| < 1$, or if $|x-2| < 1/5$; that is, in the interval $1.8 < x < 2.2$.

b) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!}$

Answer. If we rewrite this as

$$x \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \; ,$$

we see that this sums to xe^{-x^2} everywhere, so the interval of convergence is $(-\infty, \infty)$.

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5. Find the Maclaurin series for the function. **DO a) OR b).**

a) $\frac{1+x}{1-4x^2}$ b) $\int_0^x e^{-t^2} dt$

Answer. a). Start with the geometric series:

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \; .$$

Substitute $2x$ for t:

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (2x)^n \; .$$

Multiply by $1+x$:

$$\frac{1+x}{1-4x^2} = \sum_{n=0}^{\infty}(2x)^n + x\sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty}2^n x^n + \sum_{n=0}^{\infty}2^n x^{n+1} = 1 + \sum_{n=1}^{\infty}(2^n + 2^{n-1})x^n \ .$$

b) Start with the exponential series:

$$e^t = \sum_{n=0}^{\infty}\frac{x^n}{n!} \ ,$$

and substitute $-t^2$ for x:

$$e^{-t^2} = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{n!} \ ,$$

Now integrate, doing the integration on the right term by term:

$$\int_0^x e^{-t^2}dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+2)n!} \ .$$