**Calculus II**
**Exam 3, Fall 2002, Answers**

1. Find the limits

a) $\lim_{x\to e} \frac{\ln(x)-1}{\ln(\ln x)}$

**Answer**. $=^{l'H} \lim_{x\to e} \frac{\frac{1}{x}}{\frac{1}{\ln x}\frac{1}{x}} = \lim_{x\to e} \ln x = \ln e = 1$ .

b) $\lim_{x\to\infty} \frac{x(1+2x)}{3x^2+1}$

**Answer**. $= \lim_{x\to\infty} \frac{x+2x^2}{3x^2+1} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{3+\frac{1}{x^2}} = \frac{2}{3}$ .

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2. Does the integral converge or diverge? Give reasons. If you can, evaluate the integral.

a) $\int_0^1 \frac{dx}{x^{9/10}}$

**Answer**. $= \lim_{a\to 0^+} \int_a^1 x^{-9/10} dx = \lim_{a\to 0^+} 10x^{1/10}\big|_a^1 = \lim_{a\to 0^+} 10(1-a^{1/10}) = 10$ .

b) $\int_0^\infty \frac{x}{1+x^3} dx$

**Answer**. This converges because

$$\frac{x}{1+x^3} = \frac{1}{x^2+\frac{1}{x}} \le \frac{1}{x^2}$$

and

$$\int_0^\infty \frac{dx}{x^2} < \infty .$$

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3. Does the series converge or diverge? Give reasons.

a) $\sum_{n=0}^{\infty} \frac{n(2^n-1)}{3^n}$

**Answer**. This series converges by comparison with

$$\sum_{n=0}^{\infty} n(\frac{2}{3})^n$$

which is $(2/3)$ times the derived geometric series. Here is the comparison:

$$\frac{n(2^n-1)}{3^n} = n(\frac{2}{3})^n - \frac{n}{3^n} < n(\frac{2}{3})^n$$

b. $\sum_{n=0}^{\infty} \frac{1}{\ln(n)}$

**Answer**. This diverges by comparison with the series $\sum n^{-1}$, since $\ln n \le n$.

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4. What is the radius of convergence of the power series? Show your work.

a) $\sum_{n=0}^{\infty}(2^n-1)x^n$

**Answer**. We calculate the ratios of the coefficients:

$$\frac{a_{n+1}}{a_n}=\frac{2^{n+1}-1}{2^n-1}=2\frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2^n}}\to 2=\frac{1}{R}$$

so $R=1/2$.

b) $\sum_{n=0}^{\infty}\frac{3^n}{n!}x^n$

**Answer**. We calculate the ratios of the coefficients:

$$\frac{a_{n+1}}{a_n}=\frac{3^{n+1}}{(n+1)!}\frac{n!}{3^n}=\frac{3}{n+1}\to 0$$

so $R=\infty$.

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5. Find the Maclaurin series for the function.

a) $\frac{1+x}{1-x}$

**Answer**. We start with the geometric series:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

and then multiply by $1+x$:

$$\frac{1+x}{1-x}=\frac{1}{1-x}+\frac{x}{1-x}=\sum_{n=0}^{\infty}x^n+x\sum_{n=0}^{\infty}x^n=\sum_{n=0}^{\infty}x^n+\sum_{n=0}^{\infty}x^{n+1}$$

$$=\sum_{n=0}^{\infty}x^n+\sum_{n=1}^{\infty}x^n=1+2\sum_{n=1}^{\infty}x^n$$

b) $\int_0^x\frac{dt}{1-t^3}$

**Answer**. We start with the geometric series:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

and substitute $t^3$ for $x$:

$$\frac{1}{1-t^3}=\sum_{n=0}^{\infty}t^{3n}$$

and finally, integrate (term by term on the right):

$$\int_0^x\frac{dt}{1-t^3}=\sum_{n=0}^{\infty}\frac{t^{3n+1}}{3n+1}$$