Calculus II
Exam 2, Spring 2003, Answers

Find all the integrals. Remember that definite integrals should have numerical answers. You MUST show your work.

1a. $\int xe^x dx$

Answer. Integrate by parts with $u = x,\ du = dx,\ dv = e^x dx,\ v = e^x$:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C \ .$$

1b. $\int xe^{x^2} dx$

Answer. Make the substitution $u = x^2,\ du = 2xedx$:

$$\int xe^{x^2} dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C \ .$$

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2. $\int_2^4 \frac{xdx}{x^2-4}$

Answer. First, substitution works: for $u = x^2 - 4\ du = 2xdx$, we get

$$\int \frac{xdx}{x^2-4} = \frac{1}{2}\int \frac{du}{u} \ .$$

At $x = 2, u = 12$ and at $x = 2, u = 0$, so we have

$$\int_2^4 \frac{xdx}{x^2-4} = \frac{1}{2}\int_0^{12} \frac{du}{u} = \frac{1}{2}\ln u\Big|_0^{12} = \ln(12) - \ln(0) \ .$$

If you got this far you will get full credit. Since $\ln(0)$ is undefined the limit does not exist. Actually, there was a typo in the exam; the lower limit should have been 3, in which case, the answer would be ln $\sqrt{\frac{12}{5}}$.
This can also be done by the partial fractions expansion. Since $x^2 - 4 = (x-2)(x+2)$, we can write

$$\frac{x}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

for some A and B. Putting the expression on the right over a common denominator, we must have equality of the numerators: $x = A(x+2) + B(x-2)$. We find A and B by evaluating at the roots $2,\ -2$: $A = 1/2,\ B = 1/2$. Thus

$$\int_2^4 \frac{xdx}{x^2-4} = \frac{1}{2}\int_2^4 (\frac{1}{x-2} + \frac{1}{x+2})dx = \frac{1}{2}\left[\ln(x-2) + \ln(x+2)\right]_2^4 \ .$$

At this point, we have the same problem as above; since 2-2=0, we can't evaluate $\ln 0$.

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3. $\int_0^{\pi/4} \tan x \ln(\cos x) dx$

Answer. The substitution $w = \ln(\cos x),\ dw = -\tan x dx$ leads to:

$$\int_0^{\pi/4} \tan x \ln(\cos x) dx = -\int_0^{\ln(1/\sqrt{2})} w dw$$

$$= -\frac{1}{2}w^2\Big|_0^{\ln(1/\sqrt{2})} = -\frac{1}{2}(\ln(\sqrt{2}/2))^2 = -\frac{1}{8}(\ln 2)^2 \ .$$

We could also try integration by parts: if we let $dv = \ln(\cos x)dx$, we can't integrate, so we let $dv = \tan x dx$, giving $v = -\ln(\cos x)$. Then $u = \ln(\cos x),\ du = -\tan x dx$. We then get

$$\int \tan x \ln(\cos x)dx = -(\ln(\cos x))^2 - \int \tan x \ln(\cos x)dx$$

so

$$\int \tan x \ln(\cos x)dx = -\frac{1}{2}(\ln(\cos x))^2 \ .$$

Now, evaluating at the limits, we get

$$\int_0^{\pi/4} \tan x \ln(\cos x)dx = -\frac{1}{2}(\ln(\sqrt{2}/2))^2 = -\frac{1}{8}(\ln 2)^2 \ .$$

You could also remember that $\tan x = \sin x / \cos x$ suggesting the substitution

$$u = \cos x,\ du = -\sin x dx \ .$$

When $x = 0,\ u = 1$ and when $x = \pi/4,\ u = 1/\sqrt{2}$. Thus

$$\int_0^{\pi/4} \tan x \ln(\cos x)dx = -\int_0^{1/\sqrt{2}} \frac{\ln u}{u} du \ .$$

Now make the substitution $w = \ln u,\ dw = du/u$, getting

$$-\int_0^{\ln(1/\sqrt{2})} w dw = -\frac{w^2}{2}\Big|_0^{\ln(1/\sqrt{2})} \ .$$

Now, $\ln(1/\sqrt{2}) = -\ln 2/2$, so the answer is

$$\int_0^{\pi/4} \tan x \ln(\cos x)dx = -\frac{1}{2}(-\frac{1}{2}\ln 2)^2 = -\frac{1}{8}(\ln 2)^2 \ .$$

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4. $\int \frac{x}{\sqrt{1-x^2}}dx$

Answer. Let $u = 1-x^2,\ du = -2xdx$, so

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int u^{-1/2}du = -\frac{1}{2}(2u^{1/2}) + C = -\sqrt{1-x^2} + C \ .$$

We could also make the substitution $x = \sin u,\ dx = \cos u du,\ \sqrt{1-x^2} = \cos u$. We get

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int \sin u du = -\cos u + C = -\sqrt{1-x^2} + C \ .$$

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5. $\int_2^4 \frac{dx}{x(x^2-1)}$

Answer. We consider the partial fractions expansion. Since $x(x^2-1) = x(x-1)(x+1)$, we have

$$\frac{1}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1) + Bx(x+1) + Cx(x-1)}{x^2+1} \ .$$

Setting $x = 0$ and equating numerators, we get $A = -1$. For $x = 1$, we get $B = 1/2$ and for $x = -1$, we get $C = 1/2$. We can now integrate;

$$\int_2^4 \frac{dx}{x(x^2-1)} = -\int_2^4 \frac{dx}{x} + \frac{1}{2}\int_2^4 \frac{dx}{x-1} + \frac{1}{2}\int_2^4 \frac{dx}{x+1} = -\ln x + \frac{1}{2}\ln(x^2-1)\Big|_2^4$$

$$= -\ln 4 + \frac{1}{2}\ln(15) + \ln 2 - \frac{1}{2}\ln 3 = \ln(\frac{\sqrt{5}}{2}) = 1.118 \ .$$