Calculus I
Exam 1, Summer 2003, Answers

1. Solve for x:

a) $2^x = 3(5^x)$

Answer. Take logarithms of both sides. $\ln(2^x) = x\ln 2$ and $\ln(3(5^x)) = \ln 3 + x\ln 5$, so the equation becomes

$$x\ln 2 = \ln 3 + x\ln 5 \ ,$$

and the answer is

$$x = \frac{\ln 3}{\ln 2 - \ln 5} \ .$$

b) $(e^x)^5 = e^x e^3$

Answer. $(e^x)^5 = e^{5x}$ and $e^x e^3 = e^{x+3}$, so, taking logarithms, the equation becomes $5x = x+3$, which has the solution $x = 3/4$.

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2. Differentiate:

a) $f(x) = e^{2\ln x}$

Answer. $e^{2\ln x} = x^2$, so $f'(x) = 2x$. Not noticing this, you'd have:

$$f'(x) = e^{2\ln x}(\frac{2}{x}) \ ,$$

which is the same thing; nevertheless, you lost one point.

b) $g(x) = xe^x - e^x + 1$

Answer. $g'(x) = xe^x + e^x - e^x = xe^x$.

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3. A certain element decays at a rate of .000163/year. Of a piece of this element of 450 kg, how much will remain in ten years?

Answer. At the end of t years, we have $450e^{-.000163t}$ remaining. Thus, the amount after 10 years is $A = 450e^{-.00163} = 449.93$ kg.

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4. If I invest \$ 8,000 at 12.5 percent per year (compounded continuously) in how many years will my investment be worth \$ 30,000 ?

Answer. The amount I have after t years is given by $P(t) = P_0 e^{rt}$ where $P_0 = 8000$ and $r = .125$. Now for my problem, I want to find t such that $P(t) = 30000$. So, we must solve

$$30000 = 8000e^{(.125)t}$$

for t. We get $.125t = \ln(30/8) = 1.3218$, so $t = 1.3218/.125 = 10.57$ years.

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5. Solve the initial value problem $xy' + y = x,\ \ y(2) = 5$.

Answer. First solve the homogeneous equation $xy' + y = 0$, for which the variables separate: $dy/y = -dx/x$. This integrates to $\ln y = -\ln x + C = \ln(1/x) + C$, which in turn exponentiates to $y = K/x$. So, we try $y = u/x`, y' = u'/x + \cdots$ in the original equation, getting

$$xu'/x = x \quad \text{or} \quad u' = x \ ,$$

which has the solution $u = x^2/2 + C$. Thus

$$y = \frac{u}{x} = \frac{x}{2} + \frac{C}{x} \ .$$

The initial condition gives $5 = 1 + C/2$, so $C = 8$, and the answer is

$$y = \frac{x}{2} + \frac{8}{x} \ .$$