Calculus II
Exam 1, Spring 2003, Answers

1. Differentiate:

a) $f(x) = e^x \ln(x^2)$

$$f'(x) = e^x \ln(x^2) + e^x \frac{2x}{x^2} = e^x(\ln(x^2) + \frac{2}{x}) .$$

b) $g(x) = e^{3\sin(2x)}$

$$g'(x) = e^{3\sin(2x)}(3\cos(2x)(2) = 6e^{3\sin(2x)}\cos(2x) .$$

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2. Integrate:

a) $\int \frac{xe^{x^2}}{e^{x^2}+1}dx$

Let $u = e^{x^2},\ du = 2xe^{x^2}dx$, so that the integral becomes

$$\frac{1}{2}\int \frac{du}{u+1} = \frac{1}{2}\ln(u+1) + C = \frac{1}{2}\ln(e^{x^2}+1) + C .$$

b) $\int_1^3 \frac{(\ln x)^2}{x}dx$

Let $u = \ln x,\ du = dx/x$. When $x = 1,\ u = 0$ and for $x = 3$, $u = \ln 3$. The integral becomes

$$\int_0^{\ln 3} u^2 du = \frac{u^3}{3}\Big|_0^{\ln 3} = \frac{(\ln(3))^3}{3} .$$

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3. The Zombie National Bank offers accounts which pay 10.5% annually, compounded continuously. How much should I invest today so as to have \$12,000 in 6 years?

The equation for continuous growth is $P = P_0 e^{rt}$. Here $r = .105,\ t = 6,\ P = 12000$, and we are to solve for P_0. We have

$$12000 = P_0 e^{.105(6)} \quad \text{or} \quad P_0 = 12000e^{-.105(6)} = 6391.10 .$$

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4. A certain radioactive element decays so that in 100 years it has decreased to 82% its original size. What is its half-life?

Let T be the half-life of the element (in years), and r the annual rate of decay. We have the two equations

$$.82 = e^{100r} \qquad .5 = e^{rT} .$$

From the first equation $r = \ln(.82)/100$, and then the second equation becomes

$$\ln(.5) = rT = \frac{\ln(.82)}{100}T$$

giving the answer $T = 349.28$ years.

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5. Solve the initial value problem $y' + y = e^x, \; y(0) = 5$.

First solve the homogeneous equation $y' + y = 0$. This has the solution $y = Ke^{-x}$.We try $y = ue^{-x}$ in the given equation, leading to

$$u'e^{-x} = e^x \quad \text{or} \quad u' = e^{2x}$$

which has the solution $u = e^{2x}/2 + C$. Thus the general solution of our equation is

$$y = (\frac{e^{2x}}{2} + C)e^{-x} = \frac{e^x}{2} + Ce^{-x} \; .$$

The initial conditions are $y = 5$ when $x = 0$. Put that in the above equation and solve for C to get $C = 9/2$. Thus the answer is

$$y = \frac{e^x + 9e^{-x}}{2} \; .$$