

1220-90 Exam 2
Spring 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. A correct answer without accompanying work may not receive full credit. Please try to do all all work in the space provided and circle your final answer.

1. (15pts) Evaluate the following limits using L'Hôpital's rule. Note, the correct answer may be $\pm\infty$.

(a) (5pts) $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{1/(1+x)}{1} = \frac{1}{1} = 1$

5

(b) (5pts) $\lim_{x \rightarrow \infty} \frac{\sqrt[3]{x}}{\ln x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{3}x^{-2/3}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{x}{3x^{2/3}} = \lim_{x \rightarrow \infty} \frac{1}{3}x^{1/3} = +\infty$

5

(c) (5pts) $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{2\sec^2 x \tan x}{6x}$
 $\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6} = \frac{4}{6} = \frac{2}{3}$

5

2. (12pts) Evaluate the following improper integrals. Write 'divergent' if the integral diverges.

(a) (6pts) $\int_4^{\infty} \frac{1}{x^{3/2}} dx = \lim_{t \rightarrow \infty} \int_4^t x^{-3/2} dx = \lim_{t \rightarrow \infty} \left(-2x^{-1/2} \right) \Big|_4^t$
 $= \lim_{t \rightarrow \infty} \left(\frac{-2}{\sqrt{t}} + \frac{2}{\sqrt{4}} \right) = 1$

6

(b) (6pts) $\int_1^{\infty} \frac{x^3}{x^4+9} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{x^3}{x^4+9} dx$ (let $u = x^4+9$, $du = 4x^3 dx$)
 $= \lim_{t \rightarrow \infty} \frac{1}{4} \int_{10}^{t^4+9} \frac{1}{u} du = \lim_{t \rightarrow \infty} \frac{1}{4} \left(\ln|u| \right) \Big|_{10}^{t^4+9}$
 $= \lim_{t \rightarrow \infty} \frac{1}{4} \left(\ln(t^4+9) - \ln(10) \right) = +\infty$
 divergent

6

3. (8pts) Consider the sequence with first five terms given by

$$a_1 = \frac{1}{2} \quad a_2 = \frac{4}{5} \quad a_3 = \frac{9}{10} \quad a_4 = \frac{16}{17} \quad a_5 = \frac{25}{26}$$

Assuming the sequence continues on in this same manner, answer the following questions:

- (a) (2pts) Find a formula for a_n . $a_n = \frac{n^2}{n^2+1}$
- (b) (2pts) $\lim_{n \rightarrow \infty} a_n = 1$
- (c) (2pts) Is this sequence convergent or divergent? convergent
- (d) (2pts) $\sum_{n=1}^{\infty} a_n = \text{diverges}$ (write 'diverges' if the series diverges)
 (since $\lim_{n \rightarrow \infty} a_n = 1 \neq 0$)

4. (5pts) A patient is given a 30mg shot of medication every day at the same time. Immediately after the shot, the patient's blood is tested to see how much of the medication is in her bloodstream. Experiment indicates that after 24 hours $\frac{2}{5}$ of the previous day's observed amount of medicine remains in the patient's bloodstream. So, after the first shot the patient is found to have 30mg of medication in their bloodstream. The next day after the second shot, the patient is found to have 30mg of medication in their bloodstream from the second shot, plus $\frac{2}{5}$ of the previous day's amount; that makes $30 + (\frac{2}{5})(30)$ mg. After the third shot, the patient will have $30 + (\frac{2}{5})(30 + (\frac{2}{5})(30)) = 30 + 30(\frac{2}{5}) + 30(\frac{2}{5})^2$ mg. If this continues indefinitely, how much medication will the patient eventually have in her bloodstream?
 Hint: Let s_n denote the amount in the bloodstream after the n th shot. Compute $\lim_{n \rightarrow \infty} s_n$.

$$s_n = \sum_{k=1}^n (30) \left(\frac{2}{5}\right)^{k-1}$$

$$\lim_{n \rightarrow \infty} s_n = \sum_{k=1}^{\infty} (30) \left(\frac{2}{5}\right)^{k-1} \dots \text{ is a geometric series w/ } a=30 \text{ and } r=\frac{2}{5} < 1$$

$$= \frac{a}{1-r} = \frac{30}{1-\frac{2}{5}} = \frac{30}{\frac{3}{5}} = 50 \text{ mg}$$

5. (10pts) Both series below are convergent. Evaluate them.

(a) (5pts) $\sum_{n=1}^{\infty} 3 \left(\frac{5}{6}\right)^{n-1} = 3 + 3\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + \dots$
 geometric series w/ $a=3$ and $r=\frac{5}{6} < 1$. So
 $= \frac{a}{1-r} = \frac{3}{1-\frac{5}{6}} = \frac{3}{\frac{1}{6}} = 18$

(b) (5pts) $\sum_{n=3}^{\infty} \left(\frac{1}{n+5} - \frac{1}{n+6}\right) = \left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{11}\right) + \dots$
 $S_N = \sum_{n=3}^N \left(\frac{1}{n+5} - \frac{1}{n+6}\right) = \left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \dots + \left(\frac{1}{N+5} - \frac{1}{N+6}\right)$
 $= \frac{1}{8} - \frac{1}{N+6}$

So $\lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left(\frac{1}{8} - \frac{1}{N+6}\right) = \frac{1}{8}$

6. (20pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. You do not need to show work.

C D $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

C D $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 9}$

C D $\sum_{n=1}^{\infty} \sqrt[3]{2}$

C D $\sum_{n=1}^{\infty} \frac{\pi^n}{4^n}$

C D $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

C D $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

C D $\sum_{n=1}^{\infty} \frac{1}{n^5}$

C D $\sum_{n=1}^{\infty} \frac{1}{n}$

C D $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

C D $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n} \right)$

7. (10pts) Circle T if the statement is true and F if the statement is false.

T F Every absolutely convergent series is convergent.

T F Every alternating series is convergent.

T F If $\lim_{n \rightarrow \infty} a_n = 0$, the $\sum_{n=1}^{\infty} a_n$ converges.

T F If $\lim_{n \rightarrow \infty} a_n = 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

T F If $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

8. (10pts) Find the first four terms of the power series representation

$$f(x) = \frac{1}{4-x^2} = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

$$c_0 = \underline{\frac{1}{4}} \quad c_1 = \underline{0} \quad c_2 = \underline{\frac{1}{16}} \quad c_3 = \underline{0}$$

What is the radius of convergence of this power series? 2

$$\begin{aligned} \frac{1}{4-x^2} &= \frac{1}{4} \left(\frac{1}{1-\frac{x^2}{4}} \right) = \frac{1}{4} \left(1 - \left(\frac{x}{2}\right)^2 \right)^{-1} \quad \left(\text{as long as } \left| \frac{x^2}{4} \right| < 1 \text{ or } |x| < 2 \right) \\ &= \frac{1}{4} \left(\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^{2n} \right) = \frac{1}{4} \left(1 + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 + \dots \right) \\ &= \frac{1}{4} + \frac{x^2}{16} + \frac{x^4}{64} + \dots \end{aligned}$$

9. (10pts) Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x-7)^{n+1}}{3^n}$$

Use Ratio Test to test for abs. convergence: $a_n = \frac{|x-7|^{n+1}}{3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{|x-7|^{n+2}}{3^{n+1}} \cdot \frac{3^n}{|x-7|^{n+1}} = \frac{|x-7| n}{n+1}$$

$$\text{So } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{|x-7| n}{n+1} = |x-7| \lim_{n \rightarrow \infty} \frac{n}{n+1} = |x-7|$$

So series converges absolutely if $|x-7| < 1$ (the interval $(6, 8)$).

Now check endpoints:

$$\underline{x=6}: \sum_{n=1}^{\infty} \frac{(6-7)^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n} \quad \text{converges by AST}$$

$$\underline{x=8}: \sum_{n=1}^{\infty} \frac{(8-7)^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} \quad \text{diverges (p=1)}$$

So interval of convergence is $[6, 8)$
or $6 \leq x < 8$