# 1220-90 Exam 2

**Fall 2013**

Name KEY

**Instructions.** Show all work and include appropriate explanations when necessary. A correct answer without accompanying work may not receive full credit. Please try to do all all work in the space provided and circle your final answer.

1. (16pts) Evaluate the following limits using L'Hôpital's rule or any other method.

   (a) (4pts) $\lim_{x\to 0} \frac{x}{e^x}$ $= \frac{0}{e^0} = \frac{0}{1} = 0.$

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   (b) (4pts) $\lim_{x\to 0} \frac{\ln(x+1)}{\sin x}$ $\overset{L'H}{=} \lim_{x\to 0} \frac{1/(x+1)}{\cos x} = \lim_{x\to 0} \frac{1}{(x+1)\cos x} = \frac{1}{1} = 1$

   4

   (c) (4pts) $\lim_{x\to\infty} \frac{x\ln x}{x^2}$ $\overset{L'H}{=} \lim_{x\to\infty} \frac{\ln x + x\left(\frac{1}{x}\right)}{2x} = \lim_{x\to\infty} \frac{\ln x + 1}{2x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{2} = \lim_{x\to\infty} \frac{1}{2x} = 0$

   4

   (d) (4pts) $\lim_{x\to 0} \frac{\cos(2x)-1}{x^2}$ $\overset{L'H}{=} \lim_{x\to 0} \frac{-2\sin(2x)}{2x} \overset{L'H}{=} \lim_{x\to 0} \frac{-4\cos(2x)}{2} = \frac{-4}{2} = -2$

   4

2. (8pts) Consider the sequence with first five terms given by

$$a_1 = \frac{3}{2} \quad a_2 = \frac{4}{4} \quad a_3 = \frac{5}{6} \quad a_4 = \frac{6}{8} \quad a_5 = \frac{7}{10}$$

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Assuming the sequence continues on in this same manner, answer the following questions:

(a) (2pts) Find a formula for $a_n$. $a_n =$ $n+2/2n$

(b) (2pts) $\lim_{n\to\infty} a_n =$ $1/2$.

(c) (2pts) Is this sequence convergent or divergent? convergent

(d) (2pts) $\sum_{n=1}^{\infty} a_n =$ diverges (write 'diverges' if the series diverges)

Since $\lim_{n\to\infty} a_n = \frac{1}{2} > 0.$



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3. (12pts) In this problem, you will compute the value of an improper integral in two steps.

(a) (8pts) Compute the definite integral

$$\int_0^t x^2 e^{-x^3}\, dx.$$

**Note:** Your answer should be a function of $t$.

$$\int_0^t x^2 e^{-x^3} dx = -\frac{1}{3}\int_0^{-t^3} e^u du = -\frac{1}{3}\left(e^u\Big|_0^{-t^3}\right) = -\frac{1}{3}\left(e^{-t^3}-1\right)$$

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$u = -x^3$

$du = -3x^2 dx$

$$= \frac{1}{3} - \frac{1}{3}e^{-t^3}.$$

(b) (4pts) Take the limit as $t \to \infty$ of your answer in part (a) to find

$$\int_0^\infty x^2 e^{-x^3}\, dx = \lim_{t\to\infty} \int_0^t x^2 e^{-x^3}\, dx.$$

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$$\int_0^\infty x^2 e^{-x^3} dx = \lim_{t\to\infty}\left(\frac{1}{3} - \frac{1}{3}e^{-t^3}\right) = \frac{1}{3}$$

4. (7pts) Use geometric series to write the repeating decimal .474747... as a fraction.

$$.47474747\ldots = \frac{47}{100} + \frac{47}{10{,}000} + \frac{47}{1{,}000{,}000} + \cdots$$

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geometric series with $a = 47/100$, $r = \frac{1}{100}$

$$= \frac{a}{1-r} = \frac{47/100}{1-\frac{1}{100}} = 47/99$$

5. (8pts) Rewrite the following series as a collapsing series and compute its sum.

$$\sum_{n=1}^\infty \frac{2}{n^2+2n}$$

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$$\frac{2}{n^2+2n} = \frac{A}{n} + \frac{B}{n+2} = \frac{A(n+2)+Bn}{n^2+2n} \implies \begin{array}{l} A=1 \\ B=-1. \end{array}$$

So $$\sum_{n=1}^\infty \frac{2}{n^2+2n} = \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$S_N = \sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+2}\right) = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \frac{1}{5}\right) + \cdots$$
$$+ \left(\cancel{\tfrac{1}{N-1}} - \frac{1}{N+1}\right) + \left(\cancel{\tfrac{1}{N}} - \frac{1}{N+2}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

So $$\sum_{n=1}^\infty \frac{2}{n^2+2n} = \lim_{N\to\infty}\left(1+\frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) = \frac{3}{2}$$



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6. (20pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. You do not need to show work.

C D $\sum_{n=1}^{\infty} \frac{1}{n^2}$

C D $\sum_{n=1}^{\infty} \frac{1}{n5^n}$

C D $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+3)}$

C D $\sum_{n=1}^{\infty} \frac{4^n}{\pi^n}$

C D $\sum_{n=1}^{\infty} \frac{n^3}{9n^3+n^2+1}$

C D $\sum_{n=1}^{\infty} \frac{1}{n!}$

C D $\sum_{n=1}^{\infty} ne^{-n}$

C D $\sum_{n=1}^{\infty} \frac{n}{n^3+7}$

C D $\sum_{n=1}^{\infty} \cos(2\pi n)$

C D $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

7. (10pts) Find the first four terms of the power series representation

$$f(x) = \frac{6}{2+x} = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$$

$c_0 =$ 3 $c_1 =$ $-3/2$ $c_2 =$ $3/4$ $c_3 =$ $-3/8$

What is the radius of convergence of this power series? 2

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots, \quad |x| < 1.$$

$$\frac{6}{2+x} = \frac{6}{2}\left(\frac{1}{1-(-x/2)}\right) = 3\left(\frac{1}{1-(-x/2)}\right)$$

$$= 3\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n$$ as long as $\left|\frac{-x}{2}\right| < 1$ or $|x| < 2$.

$$= 3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \ldots$$



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8. (9pts) Write an expression for $a_n$ to give an example of the type of series indicated. There are many possible answers for each blank.

many possible answers.

**Example:** $\sum_{n=1}^{\infty} a_n$ is a positive series when $a_n = \frac{1}{n}$.

(a) $\sum_{n=1}^{\infty} a_n$ is an alternating series when $a_n =$ $(-1)^{n-1}$

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(b) $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent alternating series when $a_n =$ $(-1)^{n-1}/n^2$

3

(c) $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent alternating series when $a_n =$ $(-1)^{n-1}/n$

3

9. (10pts) Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n(x-2)^n}{5^n}$$

First find interval of absolute convergence:

$$a_n = \frac{n|x-2|^n}{5^n}.$$

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$$\frac{a_{n+1}}{a_n} = \frac{(n+1)|x-2|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n|x-2|^n} = \left(\frac{n+1}{n}\right)\frac{|x-2|}{5}.$$

So $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\frac{|x-2|}{5} = \frac{|x-2|}{5} < 1 \Rightarrow |x-2| < 5.$

So series converges absolutely on $|x-2|<5$ or $(-3,7)$.

Check endpoints:

$x=-3 \Rightarrow \sum_{n=1}^{\infty} n\frac{(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n n$ diverges.

$x=7 \Rightarrow \sum_{n=1}^{\infty} n\frac{(5)^n}{5^n} = \sum_{n=1}^{\infty} n$ diverges.

So interval of convergence $= (-3,7)$

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