1220-90 Exam 2

Fall 2012

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. A correct answer without accompanying work may not receive full credit. Please try to do all all work in the space provided. Please circle your final answer.

1. (16pts) Evaluate the following limits using L'Hôpital's rule or any other method.

small mistake -1
correct answer, wrong reasoning -2
something correct -3
all wrong -4
not writing limits -2 total

(a) (4pts) $\lim_{x\to 0} \frac{x}{\ln(1-x)} \overset{L'H}{=} \lim_{x\to 0} \frac{1}{\left(\frac{1}{1-x}\right)(-1)} = \lim_{x\to 0} (x-1) = -1$

(b) (4pts) $\lim_{x\to 0} \frac{\cos x - 1}{x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{-\sin x}{2x} \overset{L'H}{=} \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}$

(c) (4pts) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

(d) (4pts) $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$

2. (8pts) Consider the sequence with first five terms given by

$$a_1 = \frac{1}{3} \quad a_2 = \frac{2}{4} \quad a_3 = \frac{3}{5} \quad a_4 = \frac{4}{6} \quad a_5 = \frac{5}{7}$$

(a) (3pts) Assuming the sequence continues on in this manner, find a formula for a_n. $a_n =$ $n/n+2$

(b) (3pts) $\lim_{n\to\infty} a_n =$ 1.

(c) (2pts) Is this sequence convergent or divergent? Convergent

3. (11pts) In this problem, you will compute the value of an indefinite integral in two steps.

(a) (7pts) Use substitution to compute the definite integral

$$\int_0^t xe^{-x^2}\, dx.$$

Note: Your answer should be a function of t.

Set $u = -x^2 \Rightarrow du = -2x\,dx$ 3

$$\int_0^t xe^{-x^2}dx = -\frac{1}{2}\int_0^{-t^2} e^u du = -\frac{1}{2}\left(e^u\Big|_0^{-t^2}\right) = -\frac{1}{2}\left(e^{-t^2}-1\right)$$

2

$$= \frac{1}{2}\left(1-e^{-t^2}\right)$$

2

(b) (4pts) Take the limit as $t \to \infty$ of your answer in part (a) to find

$$\int_0^\infty xe^{-x^2}\, dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\, dx.$$

$$\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}dx = \lim_{t\to\infty}\frac{1}{2}\left(1-e^{-t^2}\right) = \frac{1}{2}.$$

3

4. (15pts) All of the following series are convergent. Evaluate them.

(a) (5pts) $\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n = \frac{4}{5} + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{5}\right)^3 + \cdots$ = Geometric series w/ $a = 4/5$ + $r = 4/5 < 1$.

recognizing geometric series (+2)

wrong a, r (−2)

Converges to +2 $\frac{a}{1-r} = \frac{4/5}{(1-4/5)} = 4.$

(b) (5pts) $\sum_{n=1}^{\infty}\frac{5}{2^{n-1}} = 5 + \frac{5}{2} + \frac{5}{2^2} + \frac{5}{2^3} + \cdots$ = Geometric series w/ $a = 5$ and $r = \frac{1}{2} < 1$

Converges to $\frac{5}{1-\frac{1}{2}} = 10.$

(c) (5pts) $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ Collapsing series 2

$$\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \left(1-\cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right) + \cdots + \left(\cancel{\tfrac{1}{N}} - \frac{1}{N+1}\right) = 1 - \frac{1}{N+1}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$$

3

5. (20pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent.

C D $\sum_{n=1}^{\infty}(-1)^n$

C D $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

C D $\sum_{n=1}^{\infty}\frac{1}{2n^3+3}$

C D $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$

C D $\sum_{n=1}^{\infty}ne^{-n^2}$

C D $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

C D $\sum_{n=1}^{\infty}\frac{n}{n+2}$

C D $\sum_{n=1}^{\infty}\frac{n}{3^n}$

C D $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$

C D $\sum_{n=1}^{\infty}\left(\frac{2}{\pi}\right)^{n-1}$

2 pts each, no p.c.

6. (10pts) Find the first four terms of the power series representation

$$f(x) = \frac{3}{(1+2x)} = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

2 pts each. wrong answers, but consistent with some valid reasoning +5

$c_0 =$ 3

$c_1 =$ -6

$c_2 =$ 12

$c_3 =$ -24

What is the radius of convergence of this power series? $|2x|<1 \Rightarrow |x|<\frac{1}{2}$. $R=\frac{1}{2}$.

$$\frac{3}{(1+2x)} = \frac{3}{(1-(-2x))} = 3\left(1+(-2x)+(-2x)^2+(-2x)^3+(-2x)^4+\cdots\right)$$

$$= 3(1-2x+4x^2-8x^3+16x^4-\cdots)$$

$$= 3-6x+12x^2-24x^3+48x^4-\cdots$$

7. (10pts) Use the integral test to determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$$

converges or diverges.

$$\int_1^\infty \frac{(\ln x)^2}{x}\,dx \quad 3 \quad = \lim_{t\to\infty} \int_1^t \frac{(\ln x)^2}{x}\,dx = \lim_{t\to\infty} \int_0^{\ln t} u^2\,du$$

$u = \ln x$
$du = \frac{1}{x}dx$ 3

$$= \lim_{t\to\infty} \left(\frac{u^3}{3} \Big|_0^{\ln t} \right) \quad 2$$

not including limits -3
correct use of IT, but incorrect integral -4

$$= \lim_{t\to\infty} \frac{(\ln t)^3}{3} = \infty.$$

So by integral test, series is divergent. 2

not necessary

Technically, we also have to check that we can use the integral test. This requires all terms to be positive (that is easy to see) and eventually decreasing. This can be seen since $\frac{d}{dx}\left(\frac{(\ln x)^2}{x}\right) = \ln x \left(\frac{2-\ln x}{x^2}\right) < 0$ when $\ln x > 2$ or $x > e^2$.

8. (10pts) Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n}$$

we use Ratio Test. 3

only giving radius of convergence $+6$

$$\frac{|x-3|^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{|x-3|^n} = \frac{|x-3|}{2}\left(\frac{n}{n+1}\right)$$

So $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x-3|}{2}\left(\frac{n}{n+1}\right) = \frac{|x-3|}{2}$

So will converge if $\frac{|x-3|}{2} < 1 \Rightarrow |x-3| < 2$. 3

Check endpoints: (2)

when $x=1$, series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n(2)^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges by AST.}$$

when $x=5$, series becomes

$$\sum_{n=1}^{\infty} \frac{(2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (p series } p=1).$$

So interval of convergence $[1,5)$ 2