1220-90 Exam 1
Spring 2014

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Please circle your final answer. The last page contains some useful identities.

1. (10 pts) Find the indicated derivatives. No need to simplify.

   (a) (5pts) $D_x(e^{x^3+x})$

   5

   $$= e^{x^3+x}\,(3x^2+1)$$

   (b) (5pts) $D_x(\ln \sqrt[3]{x})$ $= D_x(\ln x^{1/3}) = D_x\left(\frac{1}{3}\ln x\right) = \frac{1}{3x}$

   5

-1 missing +c

2. (10pts) Find the indicated antiderivatives. Remeber $+C$!!

   (a) (5pts) $\int \frac{1}{5x+2}\,dx$ $= \frac{1}{5}\int \frac{1}{u}\,du = \frac{1}{5}\ln|u| + c$

   5

   $u = 5x+2$
   $du = 5\,dx$

   $$= \frac{1}{5}\ln|5x+2| + c$$

   (b) (5pts) $\int x^2 \sin(x^3)\,dx$ $= \frac{1}{3}\int \sin(u)\,du = -\frac{1}{3}\cos(u) + c$

   5

   $u = x^3$
   $du = 3x^2\,dx$

   $$= -\frac{1}{3}\cos(x^3) + c$$

3. (6pts) Use logarithmic differentiation to find the derivative of $y = \frac{\sqrt{1+x}}{x^{7/5}}$.

   (a) (3pts) First, take the natural logarithm of the both sides and simplify using the properties of log.

   3

   $$\ln y = \ln\left(\frac{(1+x)^{1/2}}{x^{7/5}}\right) = \frac{1}{2}\ln(1+x) - \frac{7}{5}\ln x$$

   (b) (3pts) Next, differentiate both sides with respect to x and solve for $\frac{dy}{dx}$.

   3

   $$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2(1+x)} - \frac{7}{5x} \implies \frac{dy}{dx} = \left(\frac{\sqrt{1+x}}{x^{7/5}}\right)\left(\frac{1}{2(1+x)} - \frac{7}{5x}\right)$$

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4. (7pts) Let

$$f(x) = x^5 + x^3 + 1.$$

(a) (2pts) Why can we conclude that $f(x)$ has an inverse? **Hint:** Conclude something from $f'(x)$.

$f'(x) = 5x^4 + 3x^2 \geq 0 \Rightarrow f$ is monotonic $\Rightarrow f$ is 1-1.

2

2 (b) (2pts) Fill in the blank: $f^{-1}(3) =$ 1   Since $f(1) = 3$

(c) (3pts) Compute $(f^{-1})'(3)$, i.e. compute the derivative of the inverse function to f evaluated at 3.

3 $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{8}$ $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))}$

$f'(1) = 5 + 3 = 8$

5. (10pts) Evaluate the following. Any answer representing an angle should be given in radians.

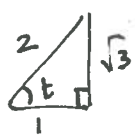
(a) $\cos^{-1}(1) =$ 0

(b) $\sin^{-1}(1) =$ $\pi/2$

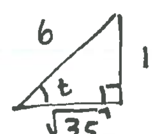
10

(c) $\cos^{-1}(\cos(3\pi)) =$ π

(d) $\sin(\cos^{-1}(\frac{1}{2})) =$ $\sqrt{3}/2$



(e) $\tan(\sin^{-1}(\frac{1}{6})) =$ $1/\sqrt{35}$



6. (10pts) According to the U.S. Census Bureau, the population of the United States was 281 million in the year 2000 and 309 million in the year 2010. Let $P(t)$ denote the population of the U.S. (in millions) t years after the year 2000. Assume the population grows exponentially.

(a) (6pts) Find constants C and k such that $P(t) = Ce^{kt}$. No need to simplify.

3 $C =$ 281 $281 = P(0) = C$

6

3 $k = \frac{1}{10}\ln\left(\frac{309}{281}\right)$

$309 = P(10) = 281e^{k(10)}$

$\frac{309}{281} = e^{k(10)} \Rightarrow k = \frac{1}{10}\ln\left(\frac{309}{281}\right)$

(b) (4pts) When will the population of the US reach 400 million? No need to simplify.

4

$400 = 281e^{\frac{t}{10}\ln(309/281)}$

$\frac{400}{281} = e^{\frac{t}{10}\ln(309/281)} \longrightarrow \ln\left(\frac{400}{281}\right) = \frac{t}{10}\ln(309/281)$

$t = \dfrac{10\ln(400/281)}{\ln(309/281)}$

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$\int u\,dv = vu - \int v\,du$

7. (10pts) Evaluate the following indefinite integrals using **integration by parts**. Remember $+C$!!

(a) $\int x \cosh x \, dx$ $= x\sinh x - \int \sinh x \, dx$

$u = x$ $du = dx$

$dv = \cosh x \, dx$ $v = \sinh x$ $= x\sinh x - \cosh x + C$

5

(b) $\int \tan^{-1} x \, dx$ **Hint:** Set $u = \tan^{-1} x$ and $dv = dx$.

$u = \tan^{-1} x$ $du = \frac{1}{1+x^2}\,dx$

5

$dv = dx$ $V = X$

$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1}x - \frac{1}{2}\int \frac{du}{u} = x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C$

$u = 1+x^2$

$du = 2x\,dx$

8. (15pts) Evaluate the following indefinite trigonometric integrals. Remember $+C$!!

(a) $\int \sin^2 x \, dx$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

5

(b) $\int \sec^4 x \tan^2 x \, dx$

$= \int \sec^2 x \tan^2 x \sec^2 x \, dx$

5

$= \int (1+\tan^2 x)\tan^2 x \sec^2 x \, dx = \int (u^2+u^4)\,du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$

$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$

$u = \tan x$

$du = \sec^2 x \, dx$

(c) $\int \sin(3x) \cos x \, dx$

$= \int \frac{1}{2}\left(\sin(2x) + \sin(4x)\right) dx = -\frac{1}{4}\cos(2x) - \frac{1}{8}\cos(4x) + C$

5

25

9. (12pts) Evaluate the following indefinite integral using **rationalizing substitution** in three steps.

$$\int x^3\sqrt{1-x^2}\,dx$$

(a) (4pts) First, write as an integral with respect to t by using the rationalizing substitution $x = \sin t$.

$x = \sin t \Rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t.$

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$dx = \cos t\,dt$

$\int x^3\sqrt{1-x^2}\,dx = \int \sin^3 t\cos^2 t\,dt$

(b) (4pts) Second, evaluate the integral from part (a). You answer should be in terms of t.

$\int \sin^3 t\cos^2 t\,dt = \int (1-\cos^2 t)\cos^2 t\sin t\,dt$

$u = \cos t$

$du = -\sin t\,dt$

4

$= -\int (u^2 - u^4)\,du = -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C = -\frac{1}{3}\cos^3 t + \frac{1}{5}\cos^5 t + C$

(c) (4pts) Finally, write you answer to part (b) in terms of x.

$x = \sin t \Rightarrow \cos t = \sqrt{1-x^2}$

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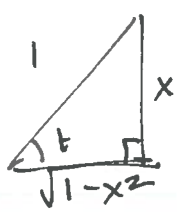

$-\frac{1}{3}\cos^3 t + \frac{1}{5}\cos^5 t + C$

$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$

10. (10pts) Use partial fractions to find $\displaystyle\int \frac{2x+2}{x^2+2x-35}\,dx$.

$\frac{2x+2}{x^2+2x-35} = \frac{A}{x+7} + \frac{B}{x-5} = \frac{A(x-5)+B(x+7)}{x^2+2x-35}$

plug in $x=5$: $12 = 12B \Rightarrow B = 1.$

10

plug in $x=-7$: $-12 = -12A \Rightarrow A = 1.$

$\int \frac{2x+2}{x^2+2x-35}\,dx = \int \left(\frac{1}{x+7} + \frac{1}{x-5}\right)dx$

$= \ln|x+7| + \ln|x-5| + C$

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