1220-90 Exam 1
Fall 2013

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Please circle your final answer. The last page contains some useful identities.

1. (25pts) Find the indicated derivatives. No need to simplify.

   (a) (5pts) $D_x(\cosh^3 x)$

   5 $= 3\cosh^2 x\,(\sinh x)$

   (b) (5pts) $D_x(e^{\sqrt[5]{x}})$ $= D_x\left(e^{x^{1/5}}\right)$

   5 $= e^{x^{1/5}} \cdot \frac{1}{5} x^{-4/5} = \dfrac{e^{\sqrt[5]{x}}}{5x^{4/5}}$

   (c) (5pts) $D_x(\ln(x^3+5))$

   5 $= \dfrac{1}{x^3+5}(3x^2) = \dfrac{3x^2}{x^3+5}$

   (d) (5pts) $D_x(\sin^{-1}(3x+5))$

   5 $= \dfrac{1}{\sqrt{1-(3x+5)^2}} \cdot 3 = \dfrac{3}{\sqrt{1-(3x+5)^2}}$

   (e) (5pts) $D_x(x^{\cos x})$ $= D_x\left(e^{\ln x \cdot \cos x}\right)$

   5 $= e^{\ln x \cos x}\left(\dfrac{\cos x}{x} - \ln x \sin x\right)$

   $= x^{\cos x}\left(\dfrac{\cos x}{x} - \ln x \sin x\right)$

25

2. (8pts) Let

$$f(x) = x^3 + 2x + 1.$$

(a) (3pts) Why can we conclude that $f(x)$ has an inverse? **Hint:** Conclude something from $f'(x)$.

3 $f'(x) = 3x^2 + 2 > 0$ ✓

since derivative is always positive, f is 1-1.

2 (b) (2pts) Fill in the blank: $f^{-1}(1) =$ 0 (since $f(0) = 1$).

(c) (3pts) Compute $(f^{-1})'(1)$, i.e. compute the derivative of the inverse function to f evaluated at 1.

3 $$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{3\cdot 0^2 + 2} = \frac{1}{2}$$

3. (10pts) Evaluate the following. Any answer representing an angle should be given in radians.

(a) $\cos^{-1}(\frac{\sqrt{3}}{2}) =$ $\pi/6$

(b) $\sin^{-1}(0) =$ 0

2 pts. each

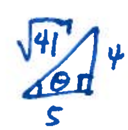
(c) $\sin(\sin^{-1}(1)) =$ 1

(d) $\cos(\sin^{-1}(\frac{1}{7})) =$ $\sqrt{48}/7$

$\theta = \sin^{-1}(\frac{1}{7}) \Longrightarrow$

(e) $\sin(\tan^{-1}(\frac{4}{5})) =$ $4/\sqrt{41}$

$\theta = \tan^{-1}(\frac{4}{5}) \Longrightarrow$



4. (10pts) A particular radioactive isotope decays from 10 grams to 7 grams in 32 hours. Let $A(t)$ denote the amount (in grams) of the isotope present after t hours.

(a) (6pts) Find constants C and k such that $A(t) = Ce^{kt}$.

6

3 $C =$ 10 $\qquad 10 = A(0) = Ce^{0t} = C.$

3 $k = \frac{1}{32}\ln(7/10)$ $\qquad 7 = 10e^{k(32)} \Rightarrow \frac{7}{10} = e^{k(32)}$

$\ln(7/10) = 32k \Rightarrow k = \frac{1}{32}\ln(7/10)$

(b) (4pts) What is the half-life of this isotope? In other words, how long will it take before half of the isotope has decayed? No need to simplify.

4

$5 = 10e^{\frac{1}{32}\ln(7/10)t}$

$\frac{1}{2} = e^{\frac{1}{32}\ln(7/10)t} \Rightarrow \ln(\frac{1}{2}) = \frac{1}{32}\ln(7/10)t$

$$t = \frac{32\ln(1/2)}{\ln(7/10)}$$

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Forgetting +C
-2 total

5. (25pts) Evaluate the following indefinite integrals. Remember $+C$!!

(a) $\int \sin^3 x \, dx$

$= \int \sin^2 x \sin x \, dx = \int (1-\cos^2 x)(\sin x) \, dx$ $\quad u = \cos x$, $du = -\sin x \, dx$

5

$= -\int (1-u^2) \, du = -u + \frac{1}{3}u^3 + C$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

(b) $\int xe^{3x} \, dx$ $\quad = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x} dx$

$u = x \quad du = dx$

$dv = e^{3x}dx \quad v = \frac{1}{3}e^{3x}$

5

$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

(c) $\int \frac{e^x}{1+9e^{2x}} \, dx$ $\quad = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1} u + C$

$u = 3e^x$

$du = 3e^x dx$

5

$= \frac{1}{3}\tan^{-1}(3e^x) + C$

(d) $\int \sin^2 x \cos^2 x \, dx$

$= \frac{1}{4}\int (1-\cos(2x))(1+\cos(2x)) = \frac{1}{4}\int (1-\cos^2(2x)) \, dx$

5

$= \frac{1}{4}\int \left(1 - \frac{1}{2}(1+\cos(4x))\right) dx = \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4x)\right) dx$

$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

(e) $\int x \ln(2x) \, dx$ $\quad = \frac{x^2}{2}\ln(2x) - \frac{1}{2}\int x \, dx$

$u = \ln(2x) \quad du = \frac{1}{x} dx$

5

$dv = x \, dx \quad v = \frac{x^2}{2}$

$= \frac{x^2}{2}\ln(2x) - \frac{x^2}{4} + C$

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6. (12pts) Evaluate the following indefinite integral using rationalizing substitution in three steps.

$$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$$

(a) (4pts) First, write as an integral with respect to t by using the rationalizing substitution $x = \tan t$.

$x = \tan t \Rightarrow dx = \sec^2 t\, dt$, $\sqrt{x^2+1} = \sqrt{\tan^2 t + 1} = \sec t$.

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$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2 t\, dt}{\tan^2 t \cdot \sec t} = \int \frac{\sec t\, dt}{\tan^2 t} = \int \frac{1/\cos t}{\sin^2/\cos^2 t}\,dt = \int \frac{\cos t}{\sin^2 t}\,dt$$

(b) (4pts) Second, evaluate the integral from part (a). You answer should be in terms of t.

4

$$\int \frac{\cos t}{\sin^2 t}\,dt = \int \frac{du}{u^2} = \int u^{-2}\,du = -u^{-1} + C$$

$u = \sin t$
$du = \cos t\, dt$

$$= \frac{-1}{\sin t} + C$$

(c) (4pts) Finally, write you answer to part (b) in terms of x.

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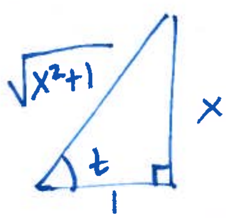
$x = \tan t \Rightarrow \tan t = \frac{x}{1}$



So $\sin t = \frac{x}{\sqrt{x^2+1}}$

$$\frac{-1}{\sin t} + C = \frac{-\sqrt{x^2+1}}{x} + C$$

7. (10pts) Use partial fractions to find $\displaystyle\int \frac{2}{x^2-3x+2}\,dx$

$$\frac{2}{(x^2-3x+2)} = \frac{A}{(x-2)} + \frac{B}{(x-1)} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)} \Rightarrow 2 = A(x-1) + B(x-2).$$

Plug in $x=1$:
$2 = -B \Rightarrow B = -2$.
Plug in $x = 2$
$2 = A \Rightarrow A = 2$.

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$$\int \frac{2}{x^2-3x+2}\,dx = \int \left(\frac{2}{x-2} - \frac{2}{x-1}\right)dx$$

$$= 2\ln|x-2| - 2\ln|x-1| + C$$

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