1220-90 Final Exam

Spring 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers. The last page contains some useful formulas.

1. (10pts) Compute the following derivatives:

   (a) (5pts) $D_x(\ln(x^2+x))$

   5

   $$= \frac{1}{x^2+x}(2x+1) = \frac{2x+1}{x^2+x}$$

   (b) (5pts) $D_x(2^{\sin^{-1} x})$

   5

   $$= 2^{\sin^{-1}x}(\ln 2) \cdot \frac{1}{\sqrt{1-x^2}}$$

-2 total

2. (10pts) Compute the following indefinite integrals: **Remember:** $+C$!

   (a) (5pts) $\int xe^{-x}\,dx$

   5

   $u = x \quad du = dx$

   $dv = e^{-x}dx \quad v = -e^{-x}$

   $$= -xe^{-x} + \int e^{-x}\,dx$$

   $$= -xe^{-x} - e^{-x} + C$$

   (b) (5pts) $\int x\ln x\,dx$

   5

   $u = \ln x \quad du = \frac{1}{x}dx$

   $dv = x\,dx \quad v = \frac{1}{2}x^2$

   $$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$$

   $$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

3. (10pts) Compute the following limits using L'Hôpital's Rule. **Note:** Answers may be $\pm\infty$.

   (a) (5pts) $\lim_{x\to 0}\frac{5x^2-\sin x}{x}$

   5

   $$\overset{L'H}{=} \lim_{x\to 0}\frac{10x-\cos x}{1} = -1$$

   (b) (5pts) $\lim_{x\to 0^+}\frac{x^2}{\sin x - x}$

   5

   $$\overset{L'H}{=} \lim_{x\to 0^+}\frac{2x}{+\cos x - 1} \overset{L'H}{=} \lim_{x\to 0^+}\frac{2}{-\sin x}$$

   $$= -\infty$$

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4. (10pts) Find the antiderivative $\int \sin^3 x \cos^4 x \, dx$.

$- 1$ missing C.

$$= \int \sin^2 x \cos^4 x \, \sin x \, dx$$

10

$$= \int (1-\cos^2 x) \cos^4 x \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int (u^4 - u^6) \, du$$

$$= -\frac{1}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

5. (10pts) Find the antiderivative $\int \frac{3}{x^2 - 5x + 6} \, dx$.

$$\frac{3}{(x^2 - 5x + 6)} = \frac{3}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$= \frac{A(x-3) + B(x-2)}{(x-2)(x-3)}$$

10

Plug in $x = 3$:

$3 = B$

Plug in $x = 2$:

$-A = 3 \Rightarrow A = -3$.

$$\int \frac{3}{x^2 - 5x + 6} \, dx = \int \left(\frac{-3}{x-2} + \frac{3}{x-3} \right) dx = -3 \ln|x-2| + 3 \ln|x-3| + C$$

4 for pf.

20

14

6. (14pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. No work is necessary.

C D $\sum_{n=1}^{\infty}(-1)^{n-1}$

C D $\sum_{n=1}^{\infty}\frac{4}{3^n}$

C D $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n}$

C D $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$

C D $\sum_{n=1}^{\infty}\frac{n!}{5^n}$

C D $\sum_{n=1}^{\infty}\frac{1}{n^4+1}$

C D $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$

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7. (12pts) Use the integral test or the comparison test to determine whether the series

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n^3+5)^2}$$

converges or diverges. **Note:** You can assume that this series satisfies the hypotheses of both tests.

Integral Test

$$\int_1^{\infty}\frac{x^2}{(2x^3+5)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{x^2}{(2x^3+5)^2}\,dx$$

$u = 2x^3+5$

$du = 6x^2\,dx$

$$= \lim_{t\to\infty}\frac{1}{6}\int_7^{2t^3+5}\frac{du}{u^2}$$

$$= \lim_{t\to\infty} -\frac{1}{6}\left(\frac{1}{u}\Big|_7^{2t^3+5}\right.$$

$$= \lim_{t\to\infty}\left(\frac{1}{42}-\frac{1}{6(2t^3+5)}\right) = \frac{1}{42}$$

So by Integral Test,

$\sum_{n=1}^{\infty}\frac{n^2}{(2n^3+5)^2}$ converges.

Comparison Test:

$$\frac{n^2}{(2n^3+5)^2} \le \frac{n^2}{(2n^3)^2} = \frac{n^2}{4n^6} = \frac{1}{4n^4}$$

Since $\sum_{n=1}^{\infty}\frac{1}{4n^4} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^4}$

converges (p-series w/ $p=4>1$), the Comparison Test says

$\sum_{n=1}^{\infty}\frac{n^2}{(2n^3+5)^2}$

converges.

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8. (10pts) Find the **radius** of convergence of the power series. **Note:** The answer may be a positive number, zero, or ∞.

$$\sum_{n=1}^{\infty} \frac{3^n(x-5)^n}{n!}.$$

✓

Use Ratio Test w/ $a_n = \dfrac{3^n|x-5|^n}{n!}$

10

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}|x-5|^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n|x-5|^n} = \frac{3|x-5|}{n+1} \to 0$$

as $n \to \infty$ for $\underline{\text{any}}$ x.

So $R = \infty$

9. (14pts) Let

$$f(x) = \frac{1}{x}$$

(a) (8pts) Find the following derivatives of $f(x)$ evaluated at $x = 1$:

8

$f(1) =$ $\underline{1}$ $\quad f(x) = x^{-1}$

$f'(1) =$ $\underline{-1}$ $\quad f'(x) = -x^{-2}$

$f''(1) =$ $\underline{2}$ $\quad f''(x) = 2x^{-3}$

$f'''(1) =$ $\underline{-6}$ $\quad f'''(x) = -6x^{-4}$

(b) (6pts) Use your computations above to write out $P_3(x)$, the Taylor polynomial of degree 3 based at $a = 1$ for $f(x)$.

6

$$P_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3$$

24

10. (10pts) Consider the conic section

$$4x^2 - y^2 - 8x - 2y = 13.$$

2

(a) (2pts) What type of conic section is this? Circle one answer below:

Parabola　　　Ellipse　　　Hyperbola

6

(b) (6pts) The center of the conic section is the point (1 , -1).

(c) (2pts) The shortest distance from the center to a point on the conic section is 2.

2

$$(4x^2 - 8x) - (y^2 + 2y) = 13$$

$$4(x^2 - 2x) - (y^2 + 2y) = 13$$

$$4(x-1)^2 - 4 - (y+1)^2 + 1 = 13$$

$$4(x-1)^2 - (y+1)^2 = 16$$

$$\frac{(x-1)^2}{(2)^2} - \frac{(y+1)^2}{4^2} = 1.$$

11. (16pts) Match the polar equation to its graph by writing the letter in the blank provided. Every answer will be used exactly once.

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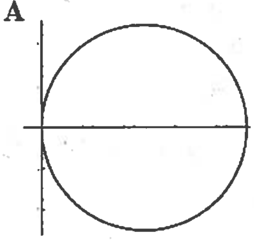
F $r = 2$
A $r = \cos\theta$
H $r = \sin\theta$
C $r = \theta$
B $r = \cos 3\theta$
E $r = \sin 2\theta$
G $r = 1 + 2\cos\theta$
D $r = 2 + 1\cos\theta$

A


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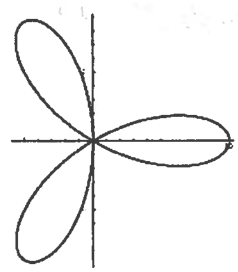

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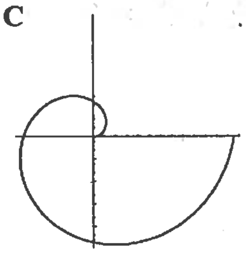

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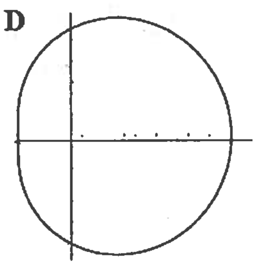

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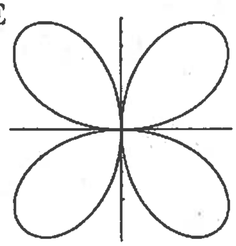

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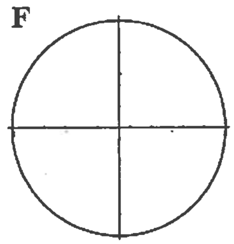

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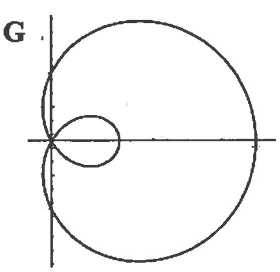

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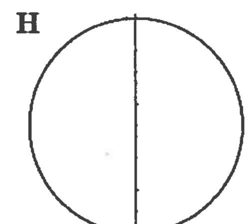


26

$^{\prime\prime} f(\theta)$

12. (24pts) Consider the cardioid determined by the polar equation $r = 2 - 2\cos\theta$. A graph of this polar curve is given a the bottom of the page.

(a) (6pts) Find the slope of the tangent line to the curve at the point $(0, 2)$ (that is, when $\theta = \frac{\pi}{2}$).

6

$$m = \frac{f'(\pi/2)\sin(\pi/2) + f(\pi/2)\cos(\pi/2)}{f'(\pi/2)\cos(\pi/2) - f(\pi/2)\sin(\pi/2)} = \frac{f'(\pi/2)}{-f(\pi/2)} = \frac{2}{-2} = -1$$

$$f(\pi/2) = 2 - 2\cos(\pi/2) = 2.$$

$$f'(\theta) = 2\sin\theta \Rightarrow f'(\pi/2) = 2$$

(b) (9pts) Find the area of the region inside the cardioid.

9

$$A = \int_0^{2\pi} \frac{1}{2}(2 - 2\cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (4 - 8\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (2 - 4\cos\theta + 2\cos^2\theta)\, d\theta$$

$$= 4\pi - 4\int_0^{2\pi} \cos\theta \, d\theta + 2\int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta)\, d\theta$$

$$= 4\pi + 2\pi = 6\pi$$

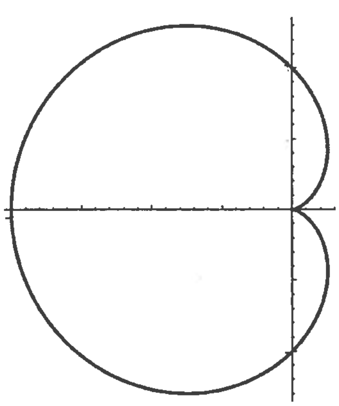
(c) (9pts) Find the perimeter of the cardioid (that is, find the arc length of $r = 2 - 2\cos\theta$ between $\theta = 0$ and $\theta = 2\pi$). **Hint:** You will have to use the identity $2\sin^2(\theta/2) = 1 - \cos\theta$.

9

$$f(\theta) = 2 - 2\cos\theta$$

$$f'(\theta) = 2\sin\theta$$

$$L = \int_0^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$



$$= \int_0^{2\pi} \sqrt{8 - 8\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{16\sin^2(\theta/2)}$$

$$= 4\int_0^{2\pi} \sin(\theta/2)\, d\theta$$

$$= 4\left(-2\cos(\theta/2)\right)\Big|_0^{2\pi}$$

$$= 8 + 8 = 16$$

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