1220-90 Final Exam
Fall 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers. The last page contains some useful formulas.

1. (10pts) Compute the following derivatives:

   (a) (5pts) $D_x(\ln(x^3 + x))$

   5

   $$= \frac{1}{x^3+x}(3x^2+1) = \frac{3x^2+1}{x^3+x}$$

   (b) (5pts) $D_x(x^{\sin x}) = D_x\left(e^{(\ln x)\sin x}\right)$

   5

   $$= e^{(\ln x)\sin x}\left(\frac{\sin x}{x} + (\ln x)\cos x\right) = x^{\sin x}\left(\frac{\sin x}{x} + \ln x \cdot \cos x\right)$$

2. (10pts) Use L'Hôpital's Rule to compute the following limits:

   (a) (5pts) $\lim_{x\to 0} \frac{e^{2x} - 1}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{2e^{2x}}{1} = 2.$

   5

   (b) (5pts) $\lim_{x\to 0} \frac{x^2}{\sin^2 x} \overset{L'H}{=} \lim_{x\to 0} \frac{2x}{2\sin x\cos x} \overset{L'H}{=} \lim_{x\to 0} \frac{2}{2\cos^2 x - 2\sin^2 x} = 1$

   5

3. (5pts) Find a formula for the inverse of the function

$$f(x) = \sqrt[3]{6-x}$$

5

$y = (6-x)^{1/3}$

$x = (6-y)^{1/3}$

$x^3 = 6 - y \Rightarrow y = 6 - x^3$

4. (24pts) Evaluate the following indefinite integrals. Remember $+C$!

(a) (6pts) $\int xe^{-3x}\,dx$ $= \frac{-x}{3}e^{-3x} + \frac{1}{3}\int e^{-3x}\,dx$

$u = x \Rightarrow du = dx$

$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$

$= \frac{-x}{3}e^{-3x} - \frac{1}{9}e^{-3x} + C$

6

(b) (6pts) $\int \sin^4 x \cos^3 x\,dx$ $= \int \sin^4 x\,(1-\sin^2 x)\cos x\,dx$

$u = \cancel{\cos x}\ \sin x$

$du = \cos x\,dx$

$= \int (u^4 - u^6)\,du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C$

6

$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

(c) (6pts) $\int \frac{1}{x^2-3x+2}\,dx$ **Hint:** $x^2-3x+2=(x-2)(x-1)$.

$\frac{1}{x^2-3x+2} = \frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)}$

when $x=1$, $1 = A\cdot 0 - B \Rightarrow B=-1$.

6 when $x=2$, $1 = A\cdot 1 + B\cdot 0 \Rightarrow A=1$

$\int \frac{1}{x^2-3x+2}\,dx = \int \left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx = \ln|x-2| - \ln|x-1| + C$

(d) (6pts) $\int \frac{1}{x^2+4x+5}\,dx$ **Hint:** $x^2+4x+5=(x+2)^2+1$.

$= \int \frac{1}{(x+2)^2+1}\,dx = \int \frac{1}{u^2+1}\,du = \tan^{-1}(u) + C$

6

$u = x+2$

$du = dx$

$= \tan^{-1}(x+2) + C$

24

5. (12pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. No work is necessary.

12

C D $\sum_{n=1}^{\infty} \frac{1}{n^3}$

C D $\sum_{n=1}^{\infty} \frac{n^2+1}{5n^2+9}$

C D $\sum_{n=1}^{\infty} 3\left(\frac{1}{8}\right)^n$

C D $\sum_{n=1}^{\infty} (-1)^{n-1} e^{-n}$

C D $\sum_{n=1}^{\infty} \sin(n\pi)$

C D $\sum_{n=1}^{\infty} \frac{n!}{n^2}$

6. (12pts) Perform the integral test in two steps:

(a) (6pts) Use **integration by parts** to find

$$\int \frac{\ln x}{x^2}\, dx.$$

Hint: Set $u = \ln x$ and $dv = x^{-2}\, dx$.

$du = \frac{1}{x} dx \qquad v = -x^{-1}$

6

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int x^{-2} dx = \frac{-\ln x}{x} - x^{-1} + C$$

$$= \frac{-\ln x}{x} - \frac{1}{x} + C$$

(b) (6pts) Use the above and the **integral test** to determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}.$$

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2} dx = \lim_{t\to\infty} \left(\frac{-\ln x}{x} - \frac{1}{x} \right)\Big|_1^t$$

6

$$= \lim_{t\to\infty} \left(-\frac{\ln t}{t} - \frac{1}{t} + 1 \right) = 0 - 0 + 1 = 1 < \infty.$$

L'Hopital's

So by Integral Test, $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges

24

7. (10pts) Find the **interval** of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2}$$

Use Ratio Test with

$a_n = \frac{|x-1|^n}{n^2}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|x-1|^n} = \lim_{n\to\infty} |x-1| \frac{n^2}{(n+1)^2} = |x-1|.$

10

So series will converge absolutely when $|x-1| < 1$ or on $(0,2)$.

7

Check endpoints: (3)

when $x=0$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by AST

when $x=2$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series $(p=2>1)$

So interval of convergence is $[0,2]$

8. (9pts) Let

$$f(x) = \ln(1+x)$$

(a) (6pts) Find the following derivatives of $f(x)$ evaluated at $x=0$:

6

$f(0) =$ 0

$f'(0) =$ 1

$f''(0) =$ −1

$f(0) = \ln(1) = 0.$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1.$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1.$

(b) (3pts) Use your computations above to write out $P_2(x)$, the second degree MacLaurin polynomial for $f(x)$.

3

$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = x - \frac{1}{2}x^2$

9. (10pts) Match the function with its Maclaurin series by writing the letter in the blank provided.

C $f(x) = e^x$

A $f(x) = \frac{1}{1-x}$

D $f(x) = x\sin x$

E $f(x) = \cos x$

B $f(x) = \frac{2}{1+2x}$

A. $\sum_{n=0}^{\infty} x^n$

B. $\sum_{n=1}^{\infty} (-1)^n 2^{n+1} x^n$

C. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

D. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

E. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

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10. (8pts) Consider the conic section

$$4x^2 + y^2 + 8x - 6y + 4 = 0.$$

2

(a) (2pts) What type of conic section is this? Circle one answer below:

Parabola Ellipse Hyperbola

4

(b) (4pts) The center of the conic section is the point (-1 , 3).

(c) (2pts) The shortest distance from the center to a point on the conic section is $3/2$.

2

$$(4x^2+8x)+(y^2-6y)+4=0$$

$$4(x^2+2x+1)-4+(y^2-6y+9)-9+4=0$$

$$4(x+1)^2+(y-3)^2=9$$

$$\frac{(x+1)^2}{(3/2)^2}+\frac{(y-3)^2}{3^2}=1$$

11. (16pts) Match the polar equation to its graph by writing the letter in the blank provided. Every answer will be used exactly once.

16

B $r = \cos 3\theta$

H $r = \sin\theta$

G $r = 1 + 2\cos\theta$

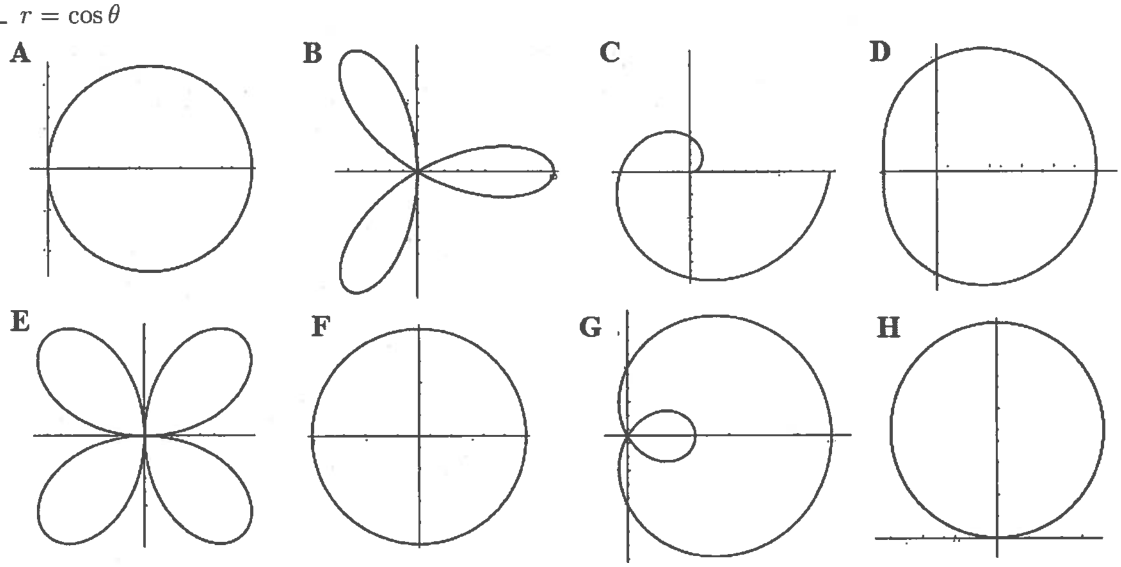
E $r = \sin 2\theta$

F $r = 2$

C $r = \theta$

D $r = 2 + 1\cos\theta$

A $r = \cos\theta$



24

12. (6pts) Match the Cartesian coordinates (x, y) on the left to the corresponding polar coordinates (r, θ) on the right by writing the letter in the blank provided. Every answer will be used exactly once.

6

B $(1, 1)$ — A. $(1, \frac{3\pi}{2})$

C $(-\sqrt{3}, 1)$ — B. $(\sqrt{2}, \frac{\pi}{4})$

A $(0, -1)$ — C. $(2, \frac{5\pi}{6})$

13. (6pts) Match the polar coordinates (r, θ) on the left to the corresponding Cartesian coordinates (x, y) on the right by writing the letter in the blank provided. Every answer will be used exactly once.

6

A $(4, \pi)$ — A. $(-4, 0)$

B $(2, \frac{4\pi}{3})$ — B. $(-1, -\sqrt{3})$

C $(\sqrt{2}, \frac{7\pi}{4})$ — C. $(1, -1)$

14. (12pts) Consider the curve determined by the polar equation $r = e^{\theta/3}$. A graph of this polar curve, between $\theta = 0$ and $\theta = \pi$ is given a the bottom of the page.

(a) (6pts) Find the area of the region inside the curve and above the x-axis.

$$A = \int_0^{\pi} \frac{1}{2}\left(e^{\theta/3}\right)^2 d\theta = \frac{1}{2}\int_0^{\pi} e^{2\theta/3}\, d\theta$$

6 ✓

$$= \frac{1}{2}\left(\frac{3}{2}\right)\left(e^{2\theta/3}\Big|_0^{\pi}\right.$$

$$= \frac{3}{4}\left(e^{2\pi/3} - 1\right)$$

(b) (6pts) Find the length of the curve.

$$f(\theta) = e^{\theta/3} \Rightarrow f'(\theta) = \frac{1}{3}e^{\theta/3}$$

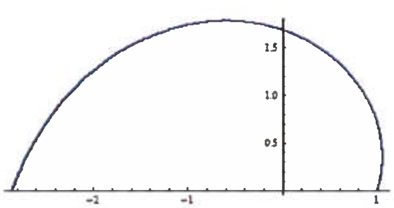
6

$$L = \int_0^{\pi} \sqrt{\left(e^{\theta/3}\right)^2 + \left(\frac{1}{3}e^{\theta/3}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{e^{2\theta/3} + \frac{1}{9}e^{2\theta/3}}\, d\theta$$

✓

$$= \int_0^{\pi} e^{\theta/3}\sqrt{1 + \frac{1}{9}}\, d\theta$$

$$= \frac{\sqrt{10}}{3}\int_0^{\pi} e^{\theta/3}\, d\theta = \frac{\sqrt{10}}{3}\left(3e^{\theta/3}\Big|_0^{\pi}\right.$$



$$= \sqrt{10}\left(e^{\pi/3} - 1\right)$$

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