Calculus I
Practice Problems 12

1. Kansas can be modelled as a rectangle of length 500 miles and of height 300 miles. The population density decreases as one moves west through the state; in fact the density x miles west of the eastern border is about

$$40 - 35 \left(\frac{x}{500}\right)^2$$

people per square mile. About what is the population of Kansas?

2. A pencil sharpener is made by drilling a cone out of a sphere; the cone has as its axis a diameter of the sphere, and its vertex is on the surface of the sphere. If the ratio of the height to base radius in the cone is 4 to 1, and the sphere has a 1 inch radius, what is the volume of the pencil sharpener?

3. An ellipsoid is formed by rotating the curve $4x^2 + y^2 = 1$ around the x-axis. What is its volume? What is the volume of the ellipsoid obtained by rotating this curve about the y-axis?

4. Consider the region in the first quadrant bounded by $y = \sin x$, $x = 0$, $x = \pi$. Find the volume of the solid obtained by rotating this region about the x-axis. Hint: $\cos(2x) = 1 - 2\sin^2 x$.

5. Consider the region in the first quadrant bounded by $y = \sin x$, $x = 0$, $x = \pi$. Find the volume of the solid obtained by rotating this region about the x-axis. Hint: The derivative of $\sin x - x\cos x$ is $x\sin x$.

6. Find the length of the curve $y = t^3$, $x = t^2$, $0 \le t \le 1$.

7. The equations $x = e^{at}\cos t$, $y = e^{at}\sin t$ define the logarithmic spiral. Find the length of this curve for $0 \le t \le 2\pi$.

8. Find the integral giving the length of the curve $y = \sqrt{1+x^2}$ for $0 \le x \le 1$.

   To find the surface area when a curve C is revolved around a line, you look at it this way: suppose a little piece of the curve, of length ds, is revolved around the line. We get a slice of a cone whose area is ds times the distanced traveled by this fragment, that is $2\pi L$, where L is the distance of the fragment from the axis of rotation. That is, $dS = 2\pi L ds$, where dS represents the differential of surface area. Now, Pappus theorem works for surface area as well for volume: if a curve is rotated about a line, it generates a surface whose area is the length of the curve times the distance travelled by the centroid of the curve.

9. Find the area of the surface obtained by rotating the line segment $y = 3x - 3$, $3 \le x \le 5$ about the x-axis.

10. Find the area of the bowl obtained by rotating the parabola $y = x^2$, $0 \le x \le a$ about the x-axis.