MATH 1210-90 Fall 2011
Third Midterm Exam
INSTRUCTOR: H.-PING HUANG

LAST NAME ____________________________
FIRST NAME ___________________________
ID NO. ________________________________

INSTRUCTION: SHOW ALL OF YOUR WORK. MAKE SURE YOUR ANSWERS ARE CLEAR AND LEGIBLE. USE SPECIFIED METHOD TO SOLVE THE QUESTION. IT IS NOT NECESSARY TO SIMPLIFY YOUR FINAL ANSWERS.

PROBLEM 1 40 _______
PROBLEM 2 20 _______
PROBLEM 3 20 _______
PROBLEM 4 20 _______
TOTAL 100 _______
PROBLEM 1

(40 pt) Analyze the function.

\[ y = f(x) = \frac{x}{1 + x^2}. \]

(3 pt) Domain and range.

Domain: all real numbers

\([-1, 1]\] is hard to determine known till sketch

(2 pt) Symmetry.

\[ f(\chi) = -f(\chi), \] odd

(3) \(x\)- and \(y\)-intercepts.

\(x: (0, 0)\) \(y: (0, 0)\)

(4) Find the first derivative of \(f\).

\[ f'(x) = \frac{1 - x^2}{(1 + x^2)^2}. \]

(5) Find the second derivative of \(f\).

\[ f''(x) = \frac{(-2x)(3 - x^2)}{(1 + x^2)^3}. \]
(6) Find the critical points, if any.

\[ 1 - x = 0 \quad x = \pm 1 \]

(7) Find the inflection points, if any.

\[ x = 0 \quad \text{or} \quad x = \pm \sqrt{\frac{3}{2}} \]

(8) Find the intervals where \( f \) is increasing, and the intervals \( f \) is decreasing.

\[ \text{inc} : (-1, 1) \]

\[ \text{dec} : (-\infty, -1) \cup (1, \infty) \]

(9) Find the intervals where \( f \) is concave up, and the intervals \( f \) is concave down.

\[ \text{up} : (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty) \]

\[ \text{down} : (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}) \]

(10) Find the asymptotes.

No vertical asym.

\[ \lim_{x \to \infty} f(x) = 0 \text{ horizontal} \]

\[ \lim_{x \to -\infty} f(x) = 0 \text{ horizontal} \]

Sketch the graph of \( f \).

\[ ( 0, \frac{\pi}{2} ) \]

\[ ( 1, \frac{1}{2} ) \]

\[ ( \sqrt{3}, \frac{\sqrt{3}}{2} ) \]

\[ (0, 0) \]

\[ (\sqrt{3}, -\sqrt{3}) \]

\[ (-1, -\frac{1}{2}) \]

\[ \text{Domain} = [-1, 1] \]

\[ (2 \text{ pt}) \]
PROBLEM 2

(20 pt) Find the dimension of the right circular cylinder of greatest volume that can be inscribed in a given right circular cone.

**Hint:** Let $a$ be the altitude and $b$ be the radius of the base of the given cone. Find out the altitude, radius, and volume, respectively, of an inscribed cylinder.

\[
\frac{a}{b} = \frac{a-h}{r}
\]

\[
r = b \left(1 - \frac{h}{a}\right)
\]

\[
V = \int b \left(1 - \frac{h}{a}\right)^2 h^2 dh
\]

\[
V = b^2 \frac{2}{2!} (1 - \frac{h}{a})^2 h
\]

\[
\frac{dV}{dh} = \pi b^2 (1 - \frac{h}{a}) (1 - \frac{3}{2} \frac{h}{a}) = 0
\]

\[
h = a \quad (r = 0) \quad (\text{meaningless})
\]

\[
h = \frac{1}{3} a, \quad r = \frac{2}{3} b, \quad V = \pi \cdot \left(\frac{3}{2}\right)^2 b \left(\frac{1}{3} a\right)
\]
PROBLEM 3

(20 pt) Use Newton’s method to find an approximation solution to the equation

\[ x^3 + x = -3 \]

as follows. Let \( x_1 = -1 \) be the initial approximation. What is the second approximation \( x_2 \)?

\[
\begin{align*}
  y &= x^3 + x + 3 \\
  y' &= 3x^2 + 1 \quad (6 \text{ pt})
\end{align*}
\]

Tangent line:

through \(( -1, 1 )\) \((5 \text{ pt})\)

\[ m = 4. \]

\[
\begin{align*}
  y &= 4(x + 1) + 1 \\
  &= 4x + 5 = 0
\end{align*}
\]

\[ x_2 = -\frac{5}{4} \quad (5 \text{ pt}) \]
PROBLEM 4

(20 pt) Consider the differential equation:

$$\frac{du}{dt} = -u^2(t^3 - t).$$

Find the particular solution of the above differential equation that satisfies the condition $u = 4$ at $t = 0$.

$$-u^{-2} \frac{du}{dt} = t^3 - t \quad (5 \text{ pt})$$

$$\int -u^{-2} \, du = \int (t^3 - t) \, dt \quad (5 \text{ pt})$$

$$u^{-1} = \frac{1}{4} t^4 - \frac{1}{2} t^2 + C \quad (5 \text{ pt})$$

$$\frac{1}{4} = C \quad (5 \text{ pt})$$

Sol: $u^{-1} = \frac{1}{4} t^4 - \frac{1}{2} t^2 + \frac{1}{4}$

or $u = \left[ \frac{1}{4} t^4 - \frac{1}{2} t^2 + \frac{1}{4} \right]^{-1}$