**Calculus I**
**Practice Exam 3, Answers**

1. Find the indefinite integral of the given function:

a) $f(x) = x^2 - 3x + x^{-2}$

**Answer**. $\int f(x)dx = \frac{x^3}{3} - \frac{3}{2}x^2 - x^{-1} + C$

b) $g(x) = \sin x + \frac{1}{\cos^2 x}$

**Answer**. $\int g(x)dx = -\cos x + \tan x + C$ since $1/\cos^2 x = \sec^2 x$.

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2. Find the function whose value at 0 is 0 and whose derivative is given.

**Answer**. Let $f(x)$ represent the answer.

a) $\frac{x}{(2x^2+1)^2}$

Make the substitution $u = 2x^2 + 1,\ du = 4xdx$. Then

$$f(x) = \int \frac{x}{(2x^2+1)^2}dx = \frac{1}{4}\int u^{-2}du = \frac{1}{4}(-\frac{1}{u}) = -\frac{1}{4(2x^2+1)} + C$$

for some constant $C$. Since $f(0) = 0$, we have $0 = -(1/4) + C$, so $C = 1/4$ and

$$f(x) = \frac{1}{4}(1 - \frac{1}{2x^2+1})\ .$$

b) $\frac{\sin x}{\cos^4 x}$

Let $u = \cos x,\ du = -\sin x dx$. Then

$$f(x) = -\int u^{-4}du = \frac{1}{3}u^{-3} + C = \frac{1}{3\cos^3 x} + C\ .$$

From the initial condition we find $C = -1/3$. Thus $f(x) = (1/3)(\cos^3 x - 1)$.

c) $\frac{\sin^2 x}{\cos^4 x}$ The integrand is $\tan^2 x \sec^2 x$, so the integral is

$$f(x) = \int \tan^2 x \sec^2 x dx = \frac{1}{3}\tan^3 x + C$$

and the initial condition gives $C = 0$.

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3. Find $y$ as a function of $x$, given that $y = 4$ when $x = 0$ and

$$\frac{dy}{dx} = x + \sin x\ .$$

**Answer**. Take the indefinite integral:

$$y = \frac{x^2}{2} - \cos x + C$$

Evaluate at (0,4) to find $C = 5$. The answer is

$$y = \frac{x^2}{2} - \cos x + 5$$

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4. Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{x}{y^2}$$

such that $y(1) = 2$.

**Answer**. Write this as an equation of differentials: $y^2 dy = x dx$, and integrate:

$$\frac{y^3}{3} = \frac{x^2}{2} + C\ .$$

Solve for $C$ by putting in the values $x = 2,\ y = 1$, to obtain $C = 13/6$. Thus the solution is given by the relation

$$\frac{y^3}{3} = \frac{x^2}{2} + \frac{13}{6}$$

which leads to the answer

$$y = (\frac{3x^2 + 13}{2})^{1/3}\ .$$

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5. Calculate the definite integrals:

a) $\int_{-4}^{4} (x^3 + 3x + \sin(2x))dx$

**Answer**. The answer is 0 since the function is an odd function, and the domain of integration is symmetric about 0.

b) $\int_{0}^{\pi/2} (\sin x \cos x)dx$

**Answer**. Let $u = \sin x,\ du = \cos x dx$. For $x = 0,\ u = 0$, and for $x = \pi/2,\ u = 1$. The integral then becomes $\int_0^1 u du$ which is 1/2.

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6. Find the definite integrals:

a) $\int_{1}^{3} x(x+1)^2 dx =$

**Answer**. $= \int_{1}^{3} (x^3 + 2x^2 + x)dx = (\frac{x^4}{4} + \frac{2}{3}x^3 + \frac{x^2}{2})|_1^3 = 41.67$ .

b) $\int_{0}^{\pi} (\sin x + \cos x)dx =$

**Answer**. $= (-\cos x + \sin x)|_0^{\pi} = 2$ .

Be careful with all the negative signs!

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7. Find the area of the region bounded by the curves $y = x^3 - x^2 + x$ and $y = x^3 + 2x^2 - 10$.

**Answer**. These are two cubic curves and we have to find their points of intersection to determine the region they enclose. Solve:

$$x^3 - x^2 + x = x^3 + 2x^2 - 10 \;, \quad \text{or} \;\; 3x^2 - x - 10 = 0 \;.$$

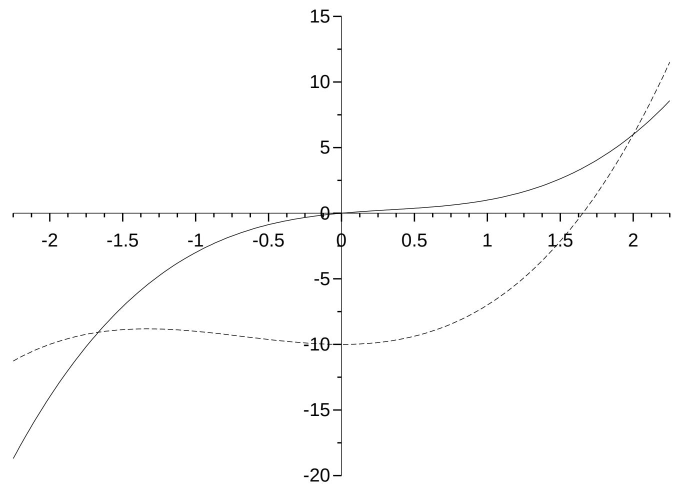
By the quadratic formula, we find

$$x = \frac{1 \pm \sqrt{1 - 4(3)(-10)}}{2(3)} = 2, \; -5/3 \;.$$

Now, evaluate the functions at some intermediate point to determine which is the upper curve, Picking 0, we get 0,-10 as the respective values (see the figure). Thus the area is

$$\int_{-5/3}^{2} (x^3 - x^2 + x - (x^3 + 2x^2 - 10))dx = (10x - \frac{x^2}{2} - x^3)\Big|_{-5/3}^{2} = \frac{1265}{64} \;.$$

The calculation disguised by the last equals sign is a long tedious piece of arithmetic (which I had Maple do for me). Remember that you also need not do this arithmetic.

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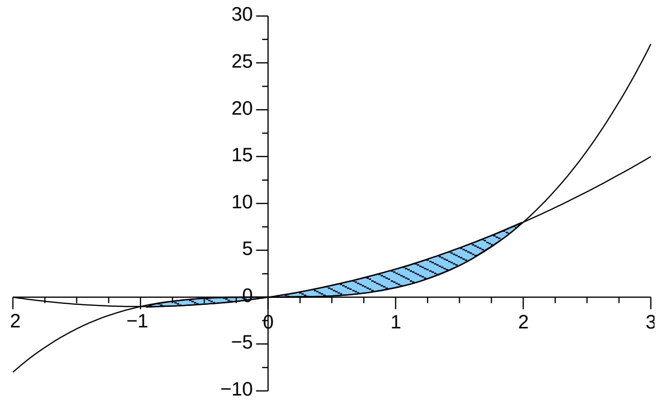
8. Find the area of the region bounded by the curves $y = x^3$ and $y = x^2 + 2x$.

**Answer**. As in problem 7, we first find the points of intersection of the two curves by solving $x^3 = x^2 + 2x$, or $x^3 - x^2 - 2x = 0$. This factors to $x(x-2)(x+1) = 0$, so the solutions are -1, 0, 2. Now the region comes in two pieces; one in the third quadrant, and the other in the first. Since, for large $x$, the cubic is above the quadratic, and they change positions at each point of intersection, we conclude that the cubic is below the quadratic in (0, 2), and above it in (-1,0) (see the figure). Thus the area is

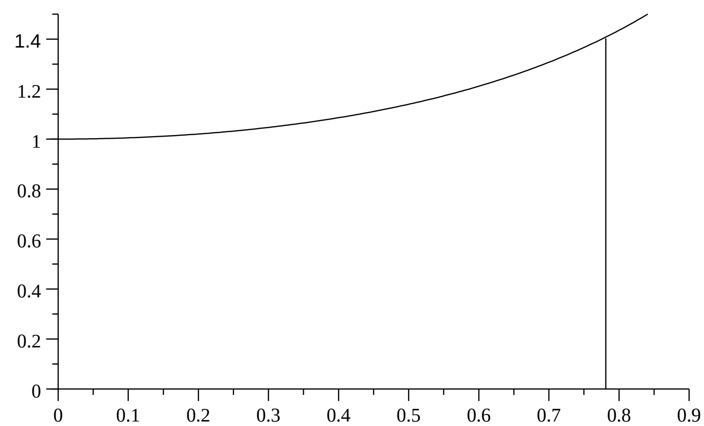
$$\int_{-1}^{0} (x^3 - (x^2 + 2x)dx + \int_{0}^{2} (x^2 + 2x - x^3)dx = \frac{5}{12} + \frac{8}{3} = \frac{37}{12} \;.$$




9. A solid lies above the region in the first quadrant bounded by the curve $y = \sec x$ from $x = 0$ to $x = \pi/4$, so that a cross-section above each line $x =$ constant is a square. What is the volume of the region?

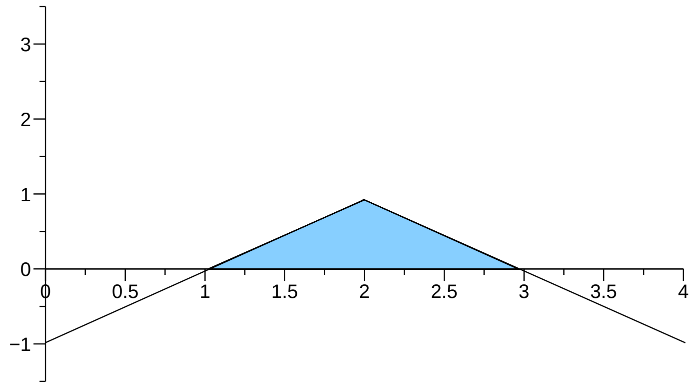
**Answer**. We calculate the volume by sweeping out the solid in the $x$ direction. For a fixed $x$, the volume of a slab of thickness $dx$ is $dV = A(x)dx$, where $A(x)$ is the area of the square of side length $\sec x$. Thus $dV = \sec^2 x dx$, and

$$V = \int_0^{\pi/4} \sec^2 x dx = \tan x \Big|_0^{\pi/4} = 1 \; .$$




10. The region in the first quadrant bounded by the curves $y = x - 1$ and $y = 3 - x$ is rotated about the $x$-axis. What is the volume of the resulting solid? What is the answer if the region is rotated about the $y$-axis?

**Answer**. Draw the picture: we get an isosceles triange with base on the $x$-axis, height 1 and vertices at $x = 1, x = 3$. We can use either the disc method (sweeping in the $x$-direction) or the shell method (sweeping in the $y$ direction).




First, the disc method: We have $dV = \pi r^2 dx$,where $r = x - 1$ from 1 to 2 and $r = 3 - x$ from 2 to 3. Thus

$$V = \int_1^2 \pi(x-1)^2 dx + \int_2^3 \pi(3-x)^2 dx \; .$$

Making the substitution $u = x - 1$ in the first integral, and $v = 3 - x$ in the second, we obtain

$$V = \pi(\int_0^1 u^2 du - \int_1^0 v^2 dv) = 2\pi \int_0^1 u^2 dx = \frac{2\pi}{3} \; .$$

Note that we could have concluded this at the beginning, by the symmetry around the line $x = 2$.
Now, the shell method. For a $y$ between 0 and 1, $dV = 2\pi y L dy$, where $L$ is the length of the line segment in the triangle at height $y$. This is the difference in the $x$ values at the endopoints which is $3 - y - (y-1) = 2 - 2y$. Thus

$$V = \int_0^1 2\pi y(2-2y)dy = 2\pi \int_0^1 (y - y^2)dy = \frac{2\pi}{3} \; .$$