Calculus I
Exam 3, Summer 2003, Answers

1. Find the Indefinite Integrals:

a) $\int (x^3 - 3x^2 + x^{-2})dx$

Answer. $\frac{x^4}{4} - x^3 - x^{-1} + C$.

b) $\int \frac{xdx}{(4x^2+1)^2}$

Answer. $\frac{1}{8}\int u^2 du = -\frac{1}{8}u^{-1} + C = -\frac{1}{8}(4x^2+1)^{-1} + C$,

using the substitution $u = 4x^2 + 1, du = 8xdx$.

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2. Find the Definite Integrals:

a) $\int_0^{\pi/2} \cos x \sin x dx$

Answer. $-\int_1^0 udu = -\frac{u^2}{2}|_1^0 = \frac{1}{2}$, using the substitution $u = \cos x, du = -\sin x dx$.(The substitution $u = \sin x$ might have been easier.)

b) $\int_0^3 (4x+1)^2 dx$

Answer. $\frac{1}{4}\int_1^{13} u^2 du = \frac{u^3}{12}|_1^{13} = \frac{13^3 - 1}{12} = 183$, using the substitution $u = 4x + 1, du = 4dx$.

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3. Find the function $y = f(x)$ which satisfies the differential equation

$$\frac{dy}{dx} = \frac{1}{yx^2}$$

such that $f(1) = 1$.

Answer. Separate the variables and integrate both sides:

$$ydy = \frac{dx}{x^2} \quad \text{so that} \quad \frac{y^2}{2} = -\frac{1}{x} + C \ .$$

Now use the initial conditions to solve for C:

$$\frac{1^2}{2} = -\frac{1}{1} + C \quad \text{so that} \quad C = \frac{3}{2} \ .$$

Finally we get

$$\frac{y^2}{2} = -\frac{1}{x} + \frac{3}{2}$$

which has the solution $y = \sqrt{3 - 2x^{-1}}$.

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4. Find the area of the region bounded by the curves $y = x + 3$ and $y = x^2 + 1$.

Answer. Draw the figure to see that the curve $y = x + 3$ is the higher curve. Find the interval of integration by finding the points of intersection of the two curves: $x + 3 = x^2 + 1$, which has the solutions $x = -1,\ 2$. Then the area is

$$Area = \int_{-1}^{2} ((x+3) - (x^2+1))dx = \int_{-1}^{2} (-x^2 + x + 2)dx = -\frac{x^3}{3} + \frac{x^2}{2} + 2x\Big|_{-1}^{2} = \frac{9}{2}$$

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5. The base of a solid is the region between the parabolas $x = y^2$ and $x = 3 - 2y^2$. Find the volume of the solid given that the cross sections perpendicular to the x-axis are squares.

Answer. These are parabolas with axis the x-axis, $x = y^2$ opening right, and $x = 3 - 2y^2$ opening left. We sweep out the volume along the x-axis, so that $dV = A(x)dx$, where $A(x)$ is the area of the square at the cross-section x. Now, x runs from 0 to 3, and $A(x) = (2y)^2$, where (x, y) lies on the parabola. Now, at some point between 0 and 3, the parabola changes from $x = y^2$ to $x = 3 - 2y^2$; that point is where the parabolas intersect. Solving the equations simultaneously, we find that point to be $(1, 1)$. Thus

$$Volume = \int_0^3 A(x)dx = \int_0^1 (2\sqrt{x})^2 dx + \int_1^3 (2\sqrt{\frac{3-x}{2}})^2 dx$$

$$= \int_0^1 4x dx + \int_1^3 2(3-x)dx = 2x^2|_0^1 + (6x - x^2)|_1^3 = 2 + ((18-9) - (6-1)) = 6\ .$$