Calculus I 1210-90 Exam 2

Summer 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. A correct answer unaccompanied by work may not receive full credit. Please try to do all work in the space provided. Please circle your final answers.

1. (13pts) For this problem, consider the function

$$f(x) = x^4 - 8x^3 + 7$$

(a) (4pts) Find $f'(x)$ and $f''(x)$.

4

2 $f'(x) = 4x^3 - 24x^2 = 4x^2(x-6)$

2 $f''(x) = 12x^2 - 48x = 12x(x-4)$

(b) (3pts) Find the two critical points of $f(x)$.

3

$0 = f'(x) = 4x^2(x-6) \implies x=0$ and $x=6$ are cps

$-$ (0) $-$ (6) $+$ sign of f'

3

(c) (3pts) Fill in the blanks: $f(x)$ is increasing on the interval (6 , $+\infty$). **Note:** $\pm\infty$ are acceptable answers.

3

(d) (3pts) Fill in the blanks: $f(x)$ is concave down on the interval (0 , 4). **Note:** $\pm\infty$ are acceptable answers.

$+$ (0) $-$ (4) $+$ sign of f''

2. (11pts) Now consider the function

$$f(x) = x^2 \cos x.$$

(a) (4pts) Show that $x = 0$ is a critical point of $f(x)$.

4

2 $f'(x) = 2x\cos x - x^2 \sin x$

2 $f'(0) = 2(0)\cos(0) - 0^2 \sin(0) = 0 - 0 = 0 \implies x=0$ is cp.

(b) (4pts) Find $f''(x)$.

4

4 $f''(x) = 2\cos x - 4x\sin x - x^2\cos x$

(c) (3pts) Use the **Second Derivative Test** to classify the critical point $x = 0$ as a local minimum, local maximum, or neither.

3

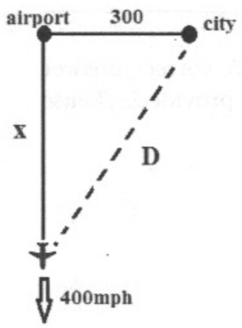
$f''(0) = 2\cos(0) - 4(0)\sin(0) - (0)^2\cos(0) = 2 - 0 - 0 = 2 > 0$

So by 2^{nd} derivative test, $x=0$ is local minimum.

24.

3. (8pts) The airport is located 300 miles due west of the city. An airplane left the airport at noon, traveling due south at 400 miles per hour. At what rate is the distance between the plane and the city (labeled D in the drawing below) increasing at 1pm?

+3 for related rates



8

$D^2 = x^2 + 300^2 = x^2 + 90000$

Now differentiate both sides w.r.t t.

$2D\frac{dD}{dt} = 2x\frac{dx}{dt}$ +2

At 1pm, $x = 400 \Rightarrow D = \sqrt{400^2+300^2} = 500$. $\frac{dx}{dt} = 400$

$2(500)\frac{dD}{dt} = 2(400)(400) \Rightarrow \frac{dD}{dt} = \frac{320000}{1000} = 320$ mph.

4. (8pts) Find the equation of the tangent line to the following curve at the point (3, 2).

$$x^2y - y^2x = 2x.$$

8

Differentiate both sides w.r.t x (assuming y is a function of x):

$2xy + x^2\frac{dy}{dx} - 2yx\frac{dy}{dx} - y^2 = 2$ +4

Plug in $x=3, y=2$:

$12 + 9\frac{dy}{dx} - 12\frac{dy}{dx} - 4 = 2 \Rightarrow -3\frac{dy}{dx} = -6 \Rightarrow \frac{dy}{dx} = 2.$

Line with slope 2 passing through (3, 2)

$y - 2 = 2(x-3)$ or $y = 2x - 4$.

5. (10pts)

(a) (6pts) Find the equation of the tangent line to $y = \sqrt[4]{x}$ at $x = 16$.

6

$f(x) = \sqrt[4]{x} = x^{1/4} \Rightarrow f(16) = 16^{1/4} = 2$

$f'(x) = \frac{1}{4}x^{-3/4} \Rightarrow f'(16) = \frac{1}{4}16^{-3/4} = \frac{1}{4}\left(\frac{1}{8}\right) = \frac{1}{32}.$

$y = 2 + \frac{1}{32}(x-16) = \frac{1}{32}x + \frac{3}{2}$

(b) (4pts) Use part (a) above to estimate $\sqrt[4]{15.9}$.

Since $15.9 \approx 16$, $\sqrt[4]{15.9} = f(15.9)$ is very close to

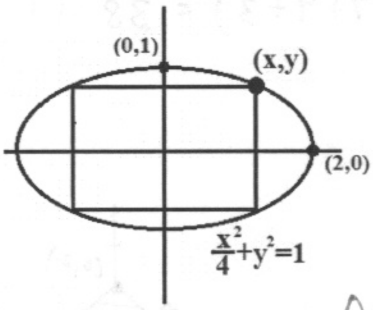
4

$2 + \frac{1}{32}(15.9-16) = 2 - \frac{.1}{32} = 2 - \frac{1}{320}$

26

6. (10pts) Find the dimensions of the rectangle with the largest area that can be inscribed in the ellipse $\frac{x^2}{4} + y^2 = 1$. Do this by finding the coordinates x and y (see picture below) that maximize the area of the rectangle squared $A = (4xy)^2 = 16x^2y^2$ (squaring the area just makes the computation easier). **Note:** You must use calculus to get credit!!

Answer: $x =$ $\sqrt{2}$ $\qquad y =$ $\frac{\sqrt{2}}{2}$



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Maximize $A = 16x^2y^2$

$\frac{x^2}{4}+y^2 = 1 \Rightarrow y^2 = 1 - \frac{x^2}{4}$.

$A(x) = 16x^2\left(1-\frac{x^2}{4}\right) = 16x^2 - 4x^4$.

Find cps:

$0 = A'(x) = 32x - 16x^3 = 16x(2-x^2) \Rightarrow x = 0$ or $x = \sqrt{2}$

($x = -\sqrt{2}$ not in first quad).

Clearly, Area is not maximized when $x = 0$.

$A''(x) = 32 - 48x^2 \Rightarrow A''(\sqrt{2}) = 32 - 48(2) < 0$

So $x = \sqrt{2}$ is a local max.

When $x = \sqrt{2}$, $y^2 = 1 - \frac{2}{4} = \frac{1}{2} \Rightarrow y = \frac{\sqrt{2}}{2}$

missing $+C$ -2

7. (16pts) Find the indicated general antiderivatives: Remember $+C$!

(a) (4pts) $\int (x-9)\, dx$

$= \frac{x^2}{2} - 9x + C$

(b) (4pts) $\int (3\cos x - \sin x)\, dx$

16

$= 3\sin x + \cos x + C$

(c) (4pts) $\int \frac{1}{x^5}\, dx$ $= \int x^{-5}\, dx$

$= -\frac{1}{4}x^{-4} + C = \frac{-1}{4x^4} + C$.

(d) (4pts) $\int (x^2+6)^8 x\, dx$

$= \frac{1}{2}\int (x^2+6)^8(2x)\,dx = \frac{1}{2}\left(\frac{1}{9}(x^2+6)^9\right) + C$

$= \frac{1}{18}(x^2+6)^9 + C$

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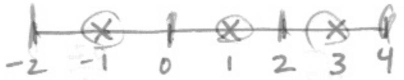
8. (6pts) Approximate $\int_{-2}^{4} (9 - x^2 + x)\, dx$ using a Riemann sum with 3 subintervals of equal length and the sample points being the midpoints of the subintervals.



$\Delta x = \frac{4-(-2)}{3} = 2.$ 2pts

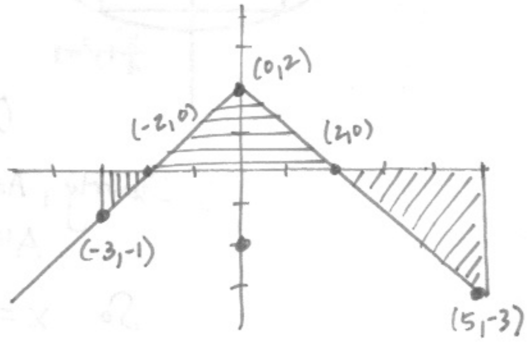
6

$\int_{-2}^{4} (9-x^2+x)\,dx \approx \Delta x\left(f(-1)+f(1)+f(3)\right) = 2(7+9+3) = 38$ 4 pts

9. (6pts) Use the area interpretation of the definite integral to evaluate

$$\int_{-3}^{5} (2 - |x|)\, dx$$

Note: It will probably be helpful to sketch a graph of $y = 2 - |x|$.



6 $\int_{-3}^{5} (2-|x|)\,dx$ = Area − Area − Area

$= \frac{1}{2}(4)(2) - \frac{1}{2}(1)(1) - \frac{1}{2}(3)(3)$

$= 4 - \frac{1}{2} - \frac{9}{2} = -1$

10. (6pts) A car's velocity in miles per hour is given by

$$v(t) = 6t^2 + 2t + 10.$$

Here, time t is measuring hours. How far has the car traveled in the first hour (from $t = 0$ to $t = 1$)?

6 Car's position is an antiderivative of $v(t)$

$s(t) = 2t^3 + t^2 + 10t + C$

Car has traveled

$s(1) - s(0) = \left(2(1)^3 + (1)^2 + 10(1) + C\right) - \left(2(0)^3 + 0^2 + 10(0) + C\right) = 13 + C - C = 13$ miles

11. (6pts) Find the value c guaranteed by the Mean Value Theorem for the function $f(x) = x^2 + 3x - 5$ on the interval $[1, 3]$.

$\frac{f(3)-f(1)}{3-1} = \frac{13-(-1)}{2} = \frac{14}{2} = 7.$

6

$7 = f'(c) = 2c + 3 \Rightarrow c = 2$

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