1210-90 Exam 2
Fall 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. A correct answer unaccompanied by work may not receive full credit. Please try to do all work in the space provided. Please circle your final answers.

1. (25pts) For this problem, consider the function

$$f(x) = x^4 - 4x^3 + 4x^2 + 1.$$

(a) (4pts) Find $f'(x)$.

4

$$f'(x) = 4x^3 - 12x^2 + 8x$$

(b) (5pts) Find the three critical points of $f(x)$.

5

$$0 = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-2)(x-1)$$

cps: $x = 0, 1, 2$

(c) (4pts) List the interval(s) on which $f(x)$ is increasing and the interval(s) on which $f(x)$ is decreasing.

4

$f'(x) = 4x(x-2)(x-1)$

$-$ 0 $+$ 1 $-$ 2 $+$ sign of f'

f is increasing on $(0,1) \cup (2,\infty)$

f is decreasing on $(-\infty,0) \cup (1,2)$

(d) (4pts) Find $f''(x)$.

4

$$f''(x) = 12x^2 - 24x + 8.$$

(e) (4pts) Use the Second Derivative Test to determine whether each critical point found in part (b) is a local minimum or a local maximum.

4

$f''(0) = 8 > 0 \Rightarrow x = 0$ is local min

$f''(1) = 12 - 24 + 8 = -4 < 0 \Rightarrow x = 1$ is a local max.

$f''(2) = 48 - 48 + 8 = 8 > 0 \Rightarrow x = 2$ is local min.

(f) (4pts) Find the maximum and minimum values of $f(x)$ on the interval $[-1, \infty)$. Write 'DNE' in the blank if there is none.

4

Maximum value= DNE

Minimum value= 1

Check values of cps in interval and "endpoints":

$f(0) = 1$

$f(1) = 1 - 4 + 4 + 1 = 2$

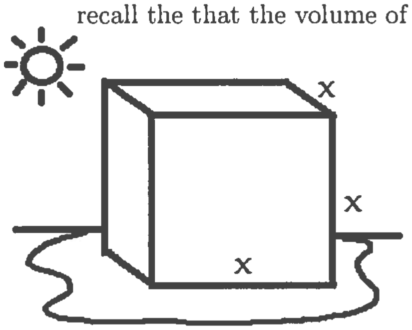
$f(2) = 16 - 32 + 16 + 1 = 1$

$f(-1) = 1 + 4 + 4 + 1 = 9$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^4 - 4x^3 + 4x^2 + 1 = +\infty.$

25

2. (8pts) An ice cube is melting in the hot sun. Suppose the ice cube is losing volume at a rate of .24 cm^3/min. How fast is the side length of the cube (labeled x in the picture below) decreasing when the volume of the ice cube is 8 cm^3? Assume that the cube remains perfectly cubical at all times and recall the that the volume of a cube is $V = x^3$.



$$V = x^3$$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$V = 8 \Rightarrow x = 2.$$

$$\frac{dV}{dt} = -.24$$

$$-.24 = 3(2)^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{-.24}{12} = -.02 \text{ cm/min}$$

3. (8pts) Find the equation of the tangent line to the following curve at the point $(1, -1)$.

$$x^2y + xy^2 + 2y = -2$$

Diff. both sides w.r.t x:

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} + 2\frac{dy}{dx} = 0.$$

Plug in $x=1,\ y=-1$

$$-2 + \frac{dy}{dx} + 1 - 2\frac{dy}{dx} + 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = 1.$$

$$y + 1 = 1(x-1) \Rightarrow y = x - 2$$

4. (5pts) Find the value c guaranteed by the Mean Value Theorem for $f(x) = x^2 + x + 3$ on the interval $[-1, 2]$

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{9-3}{3} = \frac{6}{3} = 2.$$

$$f'(x) = 2x + 1$$

$$2c + 1 = f'(c) = 2 \Rightarrow c = \tfrac{1}{2}$$

5. (5pts) Compute x_2, the second approximation to the root of

$$x^2 - 5x + 7 = 0$$

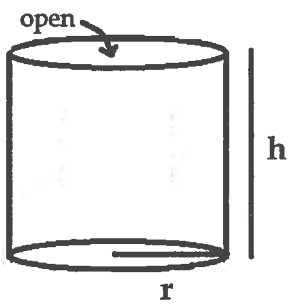
using Newton's Method with initial approximation $x_1 = 3$.

$f(x) = x^2 - 5x + 7.\quad x_1 = 3.\quad f'(x) = 2x - 5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{1}{1} = 2$$

6. (10pts) A cylindrical can with an open top is being manufactured out of 48π cm^2 of aluminum. What are the radius and height (labeled r and h in the picture below) of the can which holds the most volume? Maximize the volume of the can $V = \pi r^2 h$ subject to the fixed surface area $A = \pi r^2 + 2\pi r h$. **Note:** You must use calculus to get credit!!



maximize $V = \pi r^2 h$

$48\pi = A = \pi r^2 + 2\pi r h \Rightarrow 48\pi - \pi r^2 = 2\pi r h$

or $h = \frac{24}{r} - \frac{r}{2}$

So

$V(r) = \pi r^2\left(\frac{24}{r} - \frac{r}{2}\right) = 24\pi r - \frac{\pi r^3}{2}.$

$0 = V'(r) = 24\pi - \frac{3}{2}\pi r^2 \Longrightarrow 24 = \frac{3}{2} r^2 \Rightarrow r^2 = 16 \Rightarrow r = 4.$

Check $r=4$ is a local max $\quad V''(r) = -3\pi r \Rightarrow V''(4) = -12\pi < 0$

So by 2nd derivative test, $r = 4$ is local max.

when $r = 4$, $h = \frac{24}{4} - \frac{4}{2} = 6 - 2 = 4.$

So $r = 4, h = 4$

7. (16pts) Find the indicated general antiderivatives: Remember $+C$!

(a) (4pts) $\int (x^3 + 2x + 1)\, dx$

$= \frac{x^4}{4} + x^2 + x + C$

(b) (4pts) $\int (9 \sin x + 2)\, dx$

$= -9\cos x + 2x + C$

(c) (4pts) $\int \frac{1}{x^2}\, dx = \int x^{-2}\, dx = -x^{-1} + C = -\frac{1}{x} + C$

(d) (4pts) $\int (x^2 + 5)^9 (2x)\, dx = \frac{1}{10}(x^2+5)^{10} + C$

Since $\frac{d}{dx}(x^2+5) = 2x$, we can use "generalized power rule"

8. (12pts) The graph of $y = f(x)$ is found in Figure A below.

(a) (6pts) Approximate $\int_{-5}^{5} f(x)\,dx$ using a Riemann sum with 5 subintervals of equal length and the sample points being the midpoints of the subintervals.

6

-4 -2 0 2 4

-5 -3 -1 1 3 5

$\Delta x = \frac{5-(-5)}{5} = \frac{10}{5} = 2$

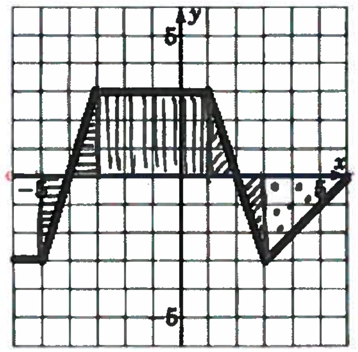


Figure A

$$\int_{-5}^{5} f(x)dx \approx \Delta x \left(f(-4)+f(-2)+f(0)+f(2)+f(4)\right)$$

$$= 2\left(0+3+3+0+-2\right)$$

$$= 8$$

(b) (6pts) Find the exact value of $\int_{-5}^{5} f(x)\,dx$. **Hint:** Use geometry and the area interpretation of the definite integral.

6

$$\int_{-5}^{5} f(x)dx = \int_{-5}^{-3} f(x)\,dx + \int_{-3}^{1} f(x)dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx$$

$$= 0 + 12 + 0 - 4 = 8$$

9. (11pts) At time $t = 0$, a driver in a car traveling at 100 feet per second applies the brakes. Suppose the car decelerates at a constant 20 feet per second squared.

(a) (4pts) Find $v(t)$, the velocity of the car in feet per second after t seconds.

4

$a(t) = -20$

$v(t) = -20t + 100$

$v(t) = \int a(t)\,dt = -20t + C$. Since $100 = v(0) \Rightarrow C = 100$.

~~[illegible]~~

(b) (3pts) How many seconds elapse before the car comes to a complete stop?

3

$0 = -20t + 100 \Rightarrow t = 5$ sec.

wbc

(c) (4pts) How many feet does the car travel before it comes to a complete stop?

4

$S(t) = \int v(t)\,dt = -10t^2 + 100t + C$.

$S(0) = 0 \Rightarrow C = 0$.

$S(t) = -10t^2 + 100t$.

$S(5) = -10(25) + 500 = 250$ ft