

COUNTING ON GRIDS¹

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In the following pages, I've collected a number of exercises related to counting on grids (or, more generally, graphs). The first set deals with problems involving binomial coefficients and introduces the first interpretation of Catalan numbers. The second set deals with a game played on the grid with two players. This leads naturally to the definition and basic properties of the permanent and determinant of an (integer) matrix. The third set of exercises provides practice with matrices, and then the fourth returns to more complicated applications of them. Next we return to the Catalan numbers and their various formulations. Finally we are led to a characterization of the Catalan numbers in terms of a certain determinant formula (whose proof follows from our earlier work on determinants).

¹A good follow-up reference for this material is the article "Counting on Determinants," by A. Benjamin and N. Cameron, V. Ponomarenko, *American Math. Monthly*, volume 112 (1995), pp. 481–492.

EXERCISES I

1. Recall that the streets of Salt Lake City (for the purposes of Math Circle) are laid out on a perfect grid. Starting at Temple Square (which is zero South and zero East), in how many ways can you travel the streets of Salt Lake City in order to arrive at the Math Department (which is 2nd South and 14th East) provided that at each stage of your journey you are traveling either South or East?

2. How many ways can you travel from Temple Square to the 9th South and 9th East Starbucks if (once again) you are allowed to only move South and East?

EXERCISES I (CONTINUED)

3. Same question: how many East-South paths are there from Temple Square to East High (13th East and 9th South).

4. How many East-South path are there from Temple Square to East High provided you are required to stop in at the 9th and 9th Starbucks for a latte?

EXERCISES I (CONTINUED)

5. How many East-South paths are there from Temple Square to my house (at 15th South and 15th East) provided you stop in at the 9th and 9th Starbucks and then meet a friend at East High before arriving?

6. How many East-South path are there from Temple Square to the City Library Drop box (at 5th South and 5th East) assuming that at each intersection of your journey your East coordinate is at least as big as your South coordinate? (For instance, you wouldn't be allowed to visit the corner of 5th South and 4th East.) Repeat the same question for the 9th and 9th Starbucks, and also for my house at 15th South and 15th East. These are difficult questions and we shall return to them in the fifth and sixth set of exercises below.

EXERCISES II

In the following set of exercises, Alice starts at 2nd South Main Street (which is zero East) and Bert starts at 2nd East South Temple (which is zero South). They travel the streets of Salt Lake City by moving only South and East. There are prizes at the following South-East coordinates: iPods are located at (4,5) and (5,4), and Xboxes are at (5,9) and (7,7). Answer the following questions:

(a) How many different East-South paths can Alice take to reach the iPod at (4,5)? How ways can she reach the other iPod?

(b) Repeat (a) for Bert.

(c) How many East-South paths can Alice and Bert travel so that they each collect an iPod? (Assume that they stop once they reach an iPod.)

(d) How many East-South paths can Alice and Bert travel so that they each collect an iPod but so that *their paths never cross*? (Assume that they stop once they reach an iPod.)

EXERCISES II (CONTINUED)

Suppose Alice and Bert each claimed an iPod and now stand, respectively, at $(4,5)$ and $(5,4)$.

(e) How many ways can Alice reach the Xbox at $(5,9)$? The one at $(7,7)$?

(f) Repeat (e) for Bert.

(g) How many East-South paths can Alice and Bert travel so that they each collect an Xbox? (Assume that they stop once they reach an Xbox.)

(h) How many East-South paths can Alice and Bert travel so that they each collect an Xbox but so that *their paths never cross*? (Assume that they stop once they reach an Xbox.)

EXERCISES II (CONTINUED)

Now assume that Alice and Bert are back at their original locations on Main Street and South Temple

(i) How many different East-South paths can Alice and Bert travel so that they each pick up an iPod and an Xbox? (Assume they stop after reaching their Xboxes.)

(j) Repeat (i) but stipulate further that their paths never cross.

EXERCISES III

Set

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \quad B = \begin{pmatrix} 6 & 4 \\ 2 & 8 \end{pmatrix}$$

Compute the following determinants and permanents:

(a) $\text{per}(A)$

(b) $\text{per}(B)$

(c) $\det(A)$

(d) $\det(B)$

(e) AB

(f) $\text{per}(AB)$

(g) $\det(AB)$

EXERCISES III (CONTINUED)

Set

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 7 & 0 \\ 3 & 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 6 & 4 & 1 \\ 2 & 8 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Compute the following determinants and permanents:

(a) $\text{per}(A)$

(b) $\text{per}(B)$

(c) $\det(A)$

(d) $\det(B)$

(e) AB

(f) $\text{per}(AB)$

(g) $\det(AB)$

EXERCISES IV

1. Now we play a new game with three players: Alice, Bert, and Chris. Alice starts at Main Street (which is zero East) and 2nd South, Bert starts at 1st East and 1st South, Chris starts at 2nd East and South Temple (which is zero South). There are prizes at the following locations: 1st East and 5th South, 3rd East and 4th South, and 5th East and 3rd South. Each player travels South and East along the streets of Salt Lake City until they reach a prize. In how many different ways can the four players each reach a prize?

2. Repeat Problem 1, but with prizes at the following locations: 2nd East and 5th South, 4th East and 4th South, and 6th East and 2nd South.

EXERCISES IV (CONTINUED)

3. David gets in on the fun and starts at 4th East and South Temple. There are now four prizes at the following East-South coordinates: $(2, 5)$, $(4, 4)$, $(5, 3)$, $(6, 2)$. How many ways can the four players each get a prize?

4. Repeat Problem 1 but this time impose the further condition that *the paths of Alice, Bert, and Chris never intersect.*

EXERCISES IV (CONTINUED)

5. Repeat Problem 2 but this time impose the further condition that *the paths of Alice, Bert, and Chris never intersect.*

6. Repeat Problem 3 but this time impose the further condition that *the paths of Alice, Bert, Chris, and David never intersect.*

EXERCISES V: CATALAN

1.(a) Compute the number of words of length 8 that contain four A 's and four B 's so that at each place in the word the number of A 's to the left is at least as large as the number of B 's. (For example, these are the possible length 6 words composed of three A 's and three B 's satisfying this condition: $ABAABB$, $ABABAB$, $AABABB$, $AABBAB$, $AAABBB$.)

(b) Compute the number of ways of inserting 3 pairs of parenthesis in the expression $1 + 2 + 3 + 4 + 5$ so as to specify the order of addition. For example the possible ways of inserting two pairs of parenthesis in $1 + 2 + 3 + 4$ are $(1 + 2) + (3 + 4)$, $(1 + (2 + 3)) + 4$, $1 + ((2 + 3) + 4)$, $((1 + 2) + 3) + 4$, $1 + (2 + (3 + 4))$

EXERCISES V: CATALAN (CONTINUED)

(c) How many ways are there of dividing a regular hexagon into triangles using non-intersecting diagonals (i.e. line segments joining two vertices). (To get started, refer to the board for the five such triangulations of the pentagon.)

(d) How many binary trees are there with four nodes? (Refer to the board for the list of binary trees with three nodes.)

EXERCISES V: CATALAN (CONTINUED)

(e) How many East-South paths are there from Temple Square to 4th South and 4th East so that at every point of the path the East coordinate is at least as large as the South coordinate?

2. Write general versions of problems 1(a)–(e) so the answers should all be the same. Why are the answers the same?

EXERCISES VI: RETURN TO CATALAN

1. Let C_n denote the number of East-South grid paths from $(0,0)$ to (n,n) such that, at each point of the path, the East coordinate is at least as large as the South coordinate. Recall that we showed

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Show that C_n also has the following interpretations.

(a) Show C_n is the number of length n words using the letters a and b so that at the each place in the word the number of a 's to the left is at least as large as the number of b 's. (For example $ababba$ is an allowable word of length 7, but $aabbbba$ is not since at the fifth place, there are three b 's to the left, but only two a 's.)

(b) Show that C_{n-1} is the number of ways of inserting $n-2$ pairs of parentheses in the expression $1+2+\cdots+n$ so as specify the order of addition. For example if $n=4$, there are 5 ways: $(1+2)+(3+4)$, $(1+(2+3))+4$, $1+((2+3)+4)$, $((1+2)+3)+4$, $1+(2+(3+4))$

(c) Show that C_n is the number of ways of dividing a regular $(n+2)$ -gon into triangles using non-intersecting diagonals (i.e. line segments joining two vertices). (To get started, draw the five such triangulations of the pentagon.)

(d) Show that C_n is the number of rooted binary trees with n nodes.

EXERCISES VII: DETERMINANTS RETURN TO CATALAN

Recall that we proved (using the “reflection principle”) that

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Consider the following matrices,

$$A_2 = \begin{pmatrix} C_0 & C_1 \\ C_1 & C_2 \end{pmatrix} \qquad A_3 = \begin{pmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{pmatrix}$$

(a) Compute $\det(A_2)$.

(b) Compute $\det(A_3)$.

(c) Generalize to the n -by- n case. (Hint: try to use the interpretation of the determinant as an appropriate number of non-intersecting paths! It turns out that the general determinant formula completely characterizes the sequence of Catalan numbers.)