

## Counting, Permutations, and Probability (continued)

**The game of MASTERMIND.** The first player chooses a secret code—a sequence of four colors (allowing repetition) from among six color types: R = red, G = green, B = blue, Y = yellow, O = orange, and P = purple. The code is represented by a sequence of colored pegs, which is hidden from the second player's view.

The second player then attempts to guess the code. He does so by placing four colored pegs on the board. After each guess, the first player reveals the following information:

1. The number of pegs which are of the right color and which are in the correct slot. (These are denoted by black pegs ●)
2. The number of pegs which are of the right color, but which are in the wrong slot. (These are denoted by ○.)

Using the information from each of the previous guesses, the second player tries to choose the 'best' guess. In such a way, the second player tries to deduce the code using as few guesses as possible.

**Coloring symmetric figures.** Suppose we are given a symmetric figure  $P$ . For instance,  $P$  might be a regular polygon or polyhedron. Suppose also that we are given  $k$  distinct colors. How many different ways can we color  $P$ ? By this, we mean that two colorings are the same if they differ by a symmetry of  $P$ .

We will focus on the problem where  $P$  is a polygon and where the goal is count the number of different ways to color the vertices of  $P$ .

For instance, if we are give two colors, say  $R$  and  $G$ , how many different ways can we color the vertices of the square? It is not too difficult to just list all the possibilities. However, we would like a method. The following method is due to Frobenius, Burnside, and Polya.

Let  $G$  be the group of symmetries of  $P$ , let  $X$  be the set of (ordered)  $k$ -colorings of the vertices of  $P$ , and let  $N$  be the number of colorings of  $P$  modulo  $G$ . So, the goal is to compute  $N$ .

**Theorem.**

$$N = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|,$$

where  $\text{Fix}(g) = \{x \in X : g(x) = x\}$ .

We need two new ideas to understand this theorem. First, the *orbit* of a coloring  $x \in X$  is the set of all colorings obtained related to  $x$  by a symmetry. Thus,

$$\text{orbit}(x) = \{g(x) : g \in G\}.$$

The *stabilizer* of a coloring  $x \in X$  is the set of all symmetries which do not change the coloring. For instance, if a coloring of 4 vertices by R and G is  $(R, G, R, R)$ , then if switching the first and third vertices is a symmetry, then this symmetry stabilizes  $(R, G, R, R)$ . We express this mathematically by

$$\text{stab}(x) = \{g \in G : g(x) = x\}.$$

**Lemma.** For each  $x \in X$ ,

$$|\text{orbit}(x)| |\text{stab}(x)| = |G|.$$

To see how this proves the theorem, consider the sum

$$\sum_{g \in G} |\text{Fix}(g)|.$$

Each  $x \in X$  appears  $|\text{stab}(x)|$  times on this list. If  $y = g(x)$ , then  $\text{stab}(y) = g\text{stab}(x)g^{-1}$ . In particular, their stabilizers have the same number of symmetries.

Now partition the terms in the sum so that colorings which are the same mod  $G$  are grouped together. Thus, we have  $|\text{orbit}(x)| |\text{stab}(x)| = |G|$  colorings in the grouping which contains  $x$ . If there are  $N$  colorings mod  $G$ , then there are  $|G|N$  terms in the sum.

Challenge: How many different ways can you  $k$ -color the cube?