Math Circle

10/15/2016

Today's problems all relate to "shuffling" numbers and sets. In mathematical language, these shufflings are referred to as *permutations*. More precisely, for a given positive integer n, a permutation σ is a map from the set $\{1, ..., n\}$ to the set $\{1, ..., n\}$ which is *one to one* (In other words, no two distinct integers in $\{1, ..., n\}$ are mapped to the same integer by σ). For example, let $n = 3$, and let

$$\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$$

be the (one-to-one) map given by

$$\sigma(1) = 2$$
$$\sigma(2) = 3$$
$$\sigma(3) = 1.$$

For any ordered pair (a, b, c) which contains each of the numbers 1,2, and 3 exactly once, we can have σ rearrange the entries:

$$(a, b, c) \mapsto (\sigma(a), \sigma(b), \sigma(c)).$$

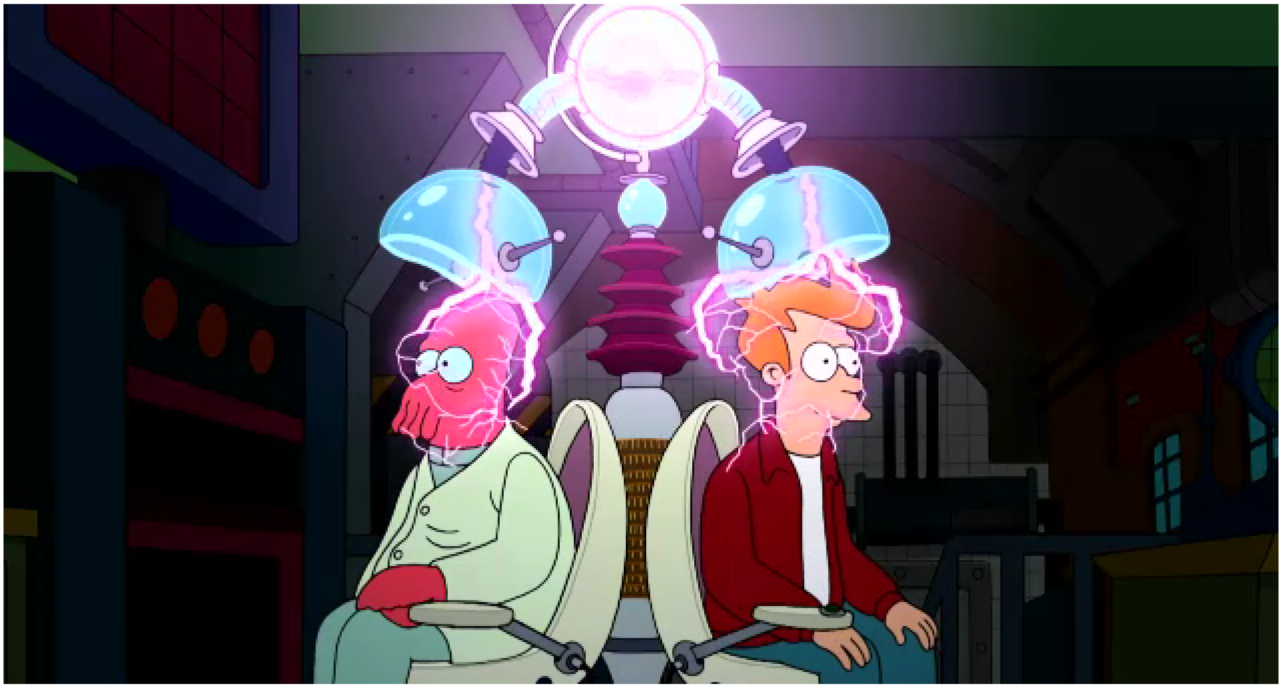
For example,

$$\sigma \cdot (2, 1, 3) = (\sigma(2), \sigma(1), \sigma(3)) = (3, 2, 1).$$

The set of all possible one-to-one maps from $\{1, 2, 3\}$ to $\{1, 2, 3\}$ is called *the symmetric group on 3 letters*. It is written as S_3 and it represents all the possible ways to shuffle a set of three things. Similarly, for any n, the set of all possible one-to-one maps from $\{1, 2, ..., n\}$ to $\{1, 2, ..., n\}$ is called the *symmetric group on n letters*. It is written as S_n and it represents all the possible ways to shuffle a set of n things.

Understanding properties of the symmetric group and properties of permutations σ is an extremely useful tool for solving problems involving arrangements of finite sets. Let's take a look at the following three problems and see how we might apply the symmetric group to solve each one:

1. The Futurama Theorem

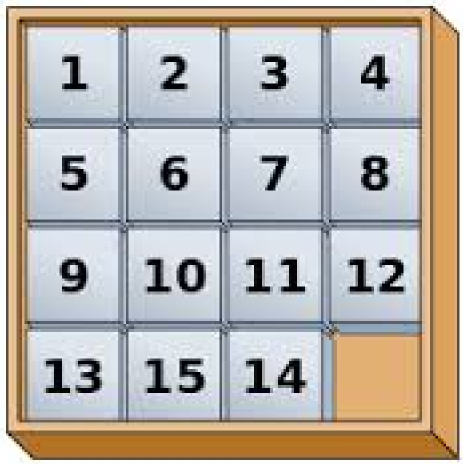


This problem involves the only mathematical theorem to ever be written for a television show! The cartoon series Futurama takes place in the year 3000; it follows the adventures of Fry, a pizza delivery boy who has been transported 1000 years in the future from 1999. In the episode "The Prisoner of Benda", Professor Farnsworth and Amy (two of Fry's coworkers) invent a machine that can switch the minds of any two people that sit inside it. Unfortunately, after the Professor and Amy use the machine, they discover that the machine can only be used once on any pair of people! As more and more characters in the show use the machine, the situation becomes more and more complicated. The final arrangement of minds and bodies is this:

$$\begin{aligned}
\text{Amy's mind} &\mapsto \text{Hermes' body}\\
\text{Professor's mind} &\mapsto \text{Bender's body}\\
\text{Bender's mind} &\mapsto \text{Emperor Nikolai's body}\\
\text{Leila's mind} &\mapsto \text{Professor's body}\\
\text{Washbucket's mind} &\mapsto \text{Amy's body}\\
\text{Fry's mind} &\mapsto \text{Zoidberg's body}\\
\text{Zoidberg's mind} &\mapsto \text{Fry's body}\\
\text{Emperor Nikolai's mind} &\mapsto \text{Washbucket's body}\\
\text{Hermes' mind} &\mapsto \text{Leila's body.}
\end{aligned}$$

The question is, is it possible to restore everyone's mind to their original bodies? Even more importantly, is it always possible to restore everyone to their original body in a situation like this?

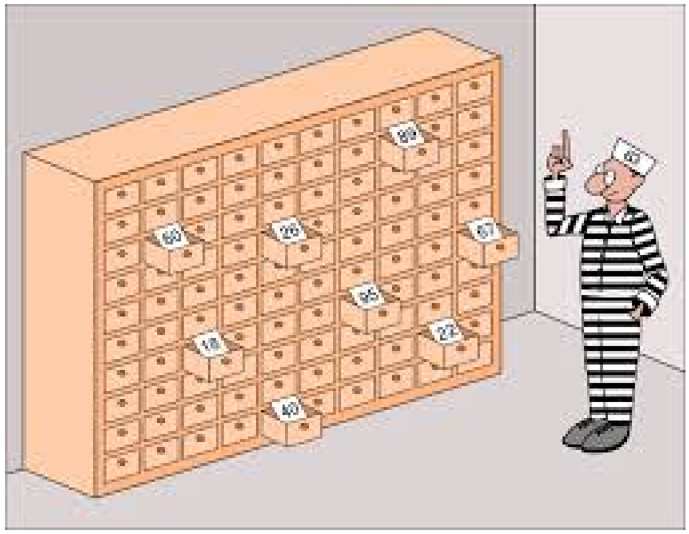
2. The 15-puzzle



The 15-puzzle is based on the sliding-piece toy shown above. In this box, the tiles may be moved by sliding them into a vacant space, but the tiles may not be lifted out of the box. For example, in the picture above, the allowable moves would be to slide either the '12' tile or the '14' tile into the lower right 'blank' space. If we slide the '12' tile into the blank space, then the allowable moves would be to slide either the '11' tile or the '8' tile into the blank space previously occupied by the '12' tile. Continuing in this way, we can shuffle around the tiles in the box; the question is, is it possible to use a sequence of tile slides to switch the '14' and the '15' tiles above while leaving all the other numbers unchanged? If so, why? If not, why?

This problem became a sensation in the 1880's when the famous chess player and puzzlemeister Sam Lloyd offered a $1000 prize to anyone who could solve the 15-puzzle. This prize money was equivalent to about $18,000 in today's money; this should be a hint as to whether or not the puzzle has a solution!

3. The 100 Prisoners Puzzle



Our last problem takes place in Mathemattica Prison! There are 100 inmates in the prison, and each prisoner wears a (distinct) number between 1 and 100. The prison warden is an amateur mathematician, and he offers the prisoners the following deal: there is cabinet in the prison storeroom that has 100 drawers. Each drawer contains a piece of paper with a distinct number between 1 and 100. The prisoners are allowed to enter the room one at a time. While the prisoner is in the room, they may open up to 50 drawers; if the number on the slip of paper in any of the 50 drawers they open matches their own, that prisoner 'wins'. If none of the numbers match their own, that prisoner 'loses'. After the prisoner leaves the room, the drawers are all closed again. The prisoners may not remove the pieces of paper from the drawers or mark them in any way, and they may not communicate with the other prisoners after they leave the storeroom. The deal is this: the prisoners are allowed to discuss a strategy beforehand; if EVERY prisoner 'wins', then all of the prisoners are set free, but if even one prisoner 'loses', then all of the prisoners remain in jail. What is the best strategy for the prisoners to use? (Surprisingly, this optimal strategy gives a 31% chance of all of the prisoners being set free!)