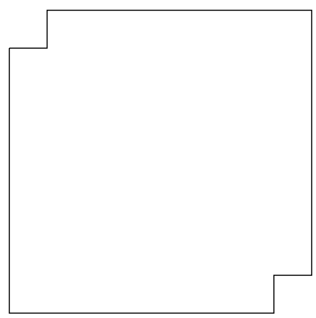
INVARIANTS AND MONOVARIANTS[1]

September 17, 2016

We will work through these problems together (mostly sequentially). If you encounter one that you've seen before, don't spoil the fun for the rest of us. Instead, skip to the next one, and work on your own or in small groups. There is enough material for at least a couple of weeks, so we will see how far we get.

0. (Warm Up) On your graph paper, draw an 8-by-8 square with two opposite one-by-one corners removed. Can this figure be tiled by 1-by-2 and 2-by-1 dominos?



1. (BMC1, Section 10.2.1, Exercise 5) Write down six 0's and five 1's. Then begin crossing out pairs of digits: either two 1's; two 0's; or a 1 and a 0. If the digits crossed out are the same, then write a new 0. If the digits crossed out are different, then write a new 1. Continue in this way until there are no more digits to cross out, and so only one digit remains. What do you see? What if we had changed the initial number of 0's and 1's?

2. (BMC1, Section 10.2.2) On a planet far away, chameleons come in three colors: green, yellow, and red. Whenever two chameleons of different colors meet, they both change to the third color.

(a) Suppore there are initially 4 green, 5 yellow, and 5 red chameleons. Is it possible to have all chameleons change to the same color?
(b) Repeat (a) for an initial configuration of 4 green, 5 yellow, and 6 red chameleons.
(c) Can you find necessary and sufficient conditions on the initial number of green, yellow, and red chameleons to be able to change into chameleons of the same color?

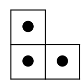
[1]Discussions of these problems—and much more—are contained in *A Decade of the Berkeley Math Circle, Volumes 1 and 2* edited by Zvezda Stankova and Tom Rike. You can find these volumes on find on amazon or at `http://bookstore.ams.org/MCL`. References to volume 1 will be given as BMC1, and similarly for BMC2. The sections referenced below are written by Sam Vandervelde and Gabriel Carroll.

3. (BMC2, Section 6.1.1, Problem 1) A rectangular $n \times m$ array of real numbers is given. Whenever the sum of the numbers in any row or column is negative, we may switch the signs of all the numbers in that row or column from negative to positive or vice-versa. Prove that if we repeat this operation enough times, eventually all the row and column sums will be non-negative.

Here is a simple 2×2 example:

$$\begin{bmatrix} -2 & -3 \\ -1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} +2 & -3 \\ +1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & +3 \\ +1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} +2 & +3 \\ -1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} +2 & -3 \\ -1 & +4 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & +3 \\ -1 & +4 \end{bmatrix} \longrightarrow \begin{bmatrix} +2 & +3 \\ +1 & +4 \end{bmatrix}$$

4. (Kontsevich's Game of Clones; BMC1, Section 5.5.3, Problem 14) Consider the following game. Draw the first quadrant of the Cartesian plane on your graph paper. Place one dot (one "clone") in each of the three bottom left-most squares.



An allowable move in the game erases one dot and replaces it with two copies in the adjacent squares, one directly above and the other directly to the right, as long as those squares are currently unoccupied. (More informally, when one clone disappears, it sprouts two new clones above and to the right of it, and two clones can't occupy the same box.) Is it possible, after a finite number of moves, to have no clones in each of the three original occupied boxes?

5. (Conway's Checkers; BMC2, Section 6.4) Imagine that you have an infinite square grid with a particular horizontal line designated. (Pull out your graph paper and draw a horizontal line across it.) You play the game as follows:

(a) First, you may initially place checkers in he squares below the line, as many as you want (but at most one per square).
(b) Then, you may take a checker and jump it over a checker that is adjacent to it (in any of the four directions) into the square immediately beyond, as long as that square is vacant. In the process, you remove the checker that has just been jumped over. Diagonal jumps are not allowed.
(c) You may continuing jumping checkers, as long as there are two checkers adjacent to each other somewhere.

The goal is to get some checker to be as far above the designated line as possible. What is the highest row that can be reached? (Start by getting one row above, then two, then three, and so on.)