

Math 2210 #6

Vector-Valued Functions and Curvilinear Motion

A vector-valued function associates a vector output, $\vec{F}(t)$, to a scalar input.
i.e $\vec{F}(t) = f(t)\hat{i} + g(t)\hat{j} = \langle f(t), g(t) \rangle$ (in 2D)
where f and g are real-valued functions of t
or (in 3D).

Definition

$\lim_{t \rightarrow c} \vec{F}(t) = \vec{L}$ means that for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that $\|\vec{F}(t) - \vec{L}\| < \varepsilon$, provided $0 < |t - c| < \delta$, i.e.

$$0 < |t - c| < \delta \Rightarrow \|\vec{F}(t) - \vec{L}\| < \varepsilon$$

Theorem A

Let $\vec{F}(t) = f(t)\hat{i} + g(t)\hat{j}$. Then \vec{F} has a limit at c iff f and g have limits at c and

$$\lim_{t \rightarrow c} \vec{F}(t) = \left[\lim_{t \rightarrow c} f(t) \right] \hat{i} + \left[\lim_{t \rightarrow c} g(t) \right] \hat{j}$$

Continuity $\Rightarrow \vec{F}(t)$ is continuous at $t = c$ if $\lim_{t \rightarrow c} \vec{F}(t) = \vec{F}(c)$.

Derivative $\Rightarrow \vec{F}'(t) = \lim_{h \rightarrow 0} \frac{\vec{F}(t+h) - \vec{F}(t)}{h}$

Differentiation Formulas

$\vec{F}(t)$ & $\vec{G}(t)$ are differentiable $c \in \mathfrak{R}$, $h(t)$ is differentiable

1. $D_t[\vec{F}(t) + \vec{G}(t)] = \vec{F}'(t) + \vec{G}'(t)$
2. $D_t[c\vec{F}(t)] = c\vec{F}'(t)$
3. $D_t[h(t) \cdot \vec{F}(t)] = h(t) \cdot \vec{F}'(t) + h'(t) \cdot \vec{F}(t)$
4. $D_t[\vec{F}(t) \cdot \vec{G}(t)] = \vec{F}(t) \cdot \vec{G}'(t) + \vec{F}'(t) \cdot \vec{G}(t)$
5. $D_t[\vec{F}(h(t))] = \vec{F}'(h(t)) \cdot h'(t)$

Integration Formula

$$\int \vec{F}(t)dt = \left[\int f(t)dt \right] \hat{i} + \left[\int g(t)dt \right] \hat{j}$$

EX 1

Find

$$\lim_{t \rightarrow \infty} \left[\frac{t \sin t}{t^2} \hat{i} - \frac{7t^3}{t^3 - 3t} \hat{j} \right]$$

EX 2

Find $\vec{F}'(x)$ and $\vec{F}''(x)$ for $\vec{F}(x) = (e^x + e^{-x^2})\hat{i} + \cos(2x)\hat{j}$.

EX 3

$$\hat{f}(y) = (\tan^2 y)\hat{i} + \sin^2(\tan^2 y)\hat{j} + 3y\hat{k}$$

Find $\vec{f}'(y)$.

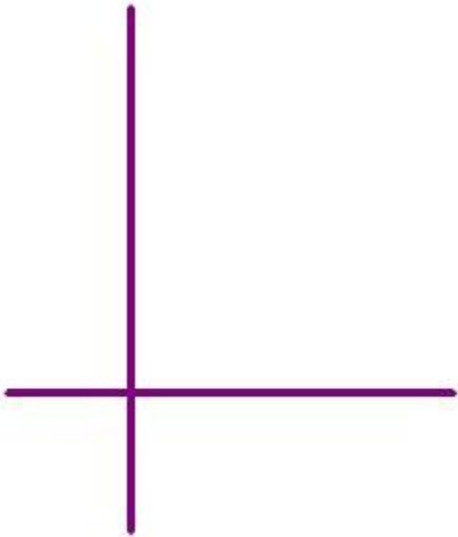
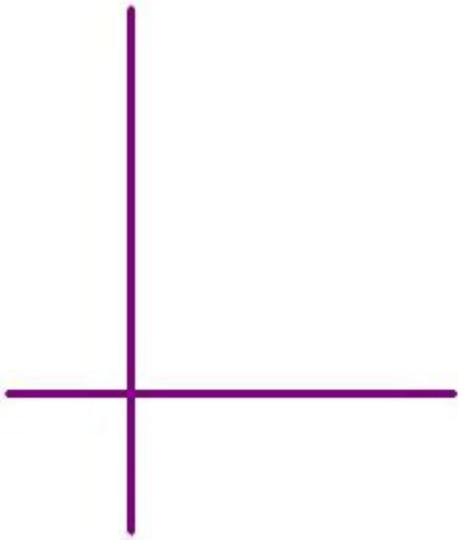
$\vec{r}(t)$ is position vector at any time t along a curve given by

$$x = x(t) \text{ and } y = y(t).$$

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

$$\vec{v}(t) = \vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j}$$

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = x''(t)\hat{i} + y''(t)\hat{j}$$



EX 4

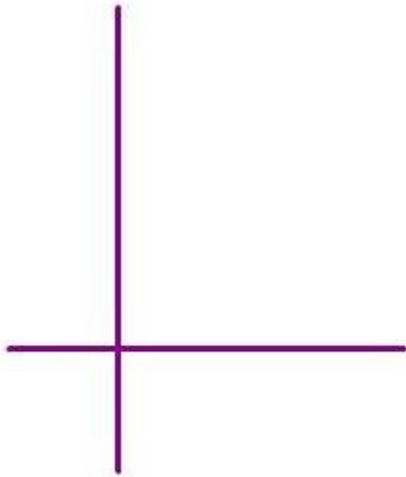
Given $\vec{r}(t) = (4\sin t)\hat{i} + (8\cos t)\hat{j}$,

4a)

Find $\vec{v}(t)$ and $\vec{a}(t)$.

4b)

Find the speed when $t = \pi/4$.

**4c)**

Sketch a portion of the graph of $\vec{r}(t)$ containing the position P of the particle at $t = \pi/4$. Draw \vec{v} and \vec{a} at P as well.

EX 5

Suppose that an object moves around a circle with center at $(0,0)$ and radius r at a constant angular speed of ω radians/sec. If its initial position is $(0, r)$, find its acceleration.