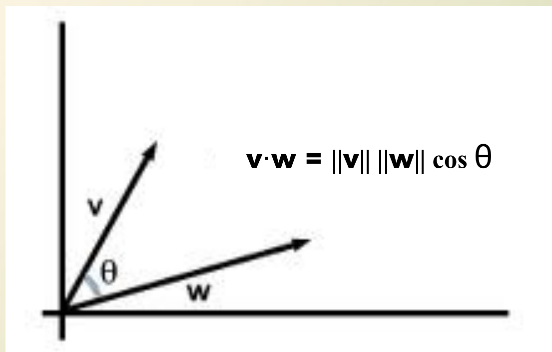
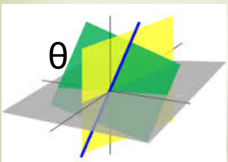
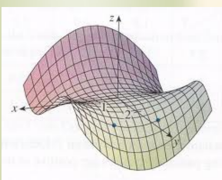
The Dot Product





$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy = \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy = \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

EX 1

a) Write a vector represented by $\overrightarrow{AB}$ in the form $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

A (-2, 3, 5) B (1, -2, 4)

$\vec{a} = (1-(-2))\hat{i} + (-2-3)\hat{j} + (4-5)\hat{k}$

$= 3\hat{i} - 5\hat{j} + -1\hat{k}$ (or $\langle 3, -5, -1 \rangle$)

b) Find a unit vector $\vec{u}$ in the direction of ⟨-3, 5, 6⟩ and express it in the form $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$

$\hat{u} = \dfrac{\vec{u}}{\|\vec{u}\|}$ $\|\vec{u}\| = \sqrt{3^2+5^2+6^2} = \sqrt{9+25+36}$

$= \sqrt{70}$

$\hat{u} = \dfrac{\langle -3, 5, 6 \rangle}{\sqrt{70}} = \dfrac{-3}{\sqrt{70}}\hat{i} + \dfrac{5}{\sqrt{70}}\hat{j} + \dfrac{6}{\sqrt{70}}\hat{k}$

The dot product is one type of multiplication between vectors that returns a scalar (number).

For $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$,

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}\ .$$

Theorem A

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and c a real number.

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutativity of dot product)

$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributivity)

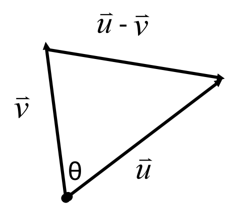
$c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$ (associativity/commutativity of scalar multiplication)

$\vec{0} \cdot \vec{u} = 0$

$\boxed{\vec{u} \cdot \vec{u} = \| \vec{u} \|^2}$

Theorem B

Geometrically, we can think of the dot product as $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.



A use law of Cosines

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\|\vec{v}\|\|\vec{u}\|\cos\theta$$

But also, we know B

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$
$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$
$$= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

equate A and B

$$\cancel{\|\vec{u}\|^2} - 2\vec{u}\cdot\vec{v} + \cancel{\|\vec{v}\|^2} = \cancel{\|\vec{v}\|^2} + \cancel{\|\vec{u}\|^2} - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$-2\vec{u}\cdot\vec{v} = -2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta}$$

notice:
$$\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \cos\theta$$
$$\left(\frac{\vec{u}}{\|\vec{u}\|}\right)\cdot\left(\frac{\vec{v}}{\|\vec{v}\|}\right) = \cos\theta$$
$$\hat{u}\cdot\hat{v} = \cos\theta$$

Theorem C

The vectors $\vec{u}$ and $\vec{v}$ are perpendicular iff $\vec{u} \cdot \vec{v} = 0$.

Perpendicular vectors are called orthogonal.

if $\vec{u} \cdot \vec{v} = 0$

$\Leftrightarrow \|\vec{u}\| \|\vec{v}\| \cos\theta = 0$

$\cos\theta = 0$

$\theta = \pi/2$

EX 2 For what number c are these vectors perpendicular?

$\langle 2c, -8, 1 \rangle$ and $\langle 3, c, -2+c \rangle$

orthog. if $\langle 2c, -8, 1 \rangle \cdot \langle 3, c, -2+c \rangle = 0$

$2c(3) + -8(c) + 1(-2+c) = 0$

$6c - 8c - 2 + c = 0$

$-c = 2$

$\boxed{c = -2}$

EX 3 Find the angle between $\vec{u} = 4\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{v} = 2\hat{i} + \hat{j} + 5\hat{k}$.

$$\vec{u} \cdot \vec{v} = \langle 4, 2, 3 \rangle \cdot \langle 2, 1, 5 \rangle$$
$$= 4(2) + 2(1) + 3(5)$$
$$= 25$$

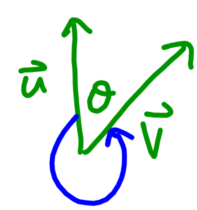
$\vec{u} \cdot \vec{v}$
$= u_1 v_1 + u_2 v_2 + u_3 v_3$
where
$\vec{u} = \langle u_1, u_2, u_3 \rangle$

also $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$ (*)

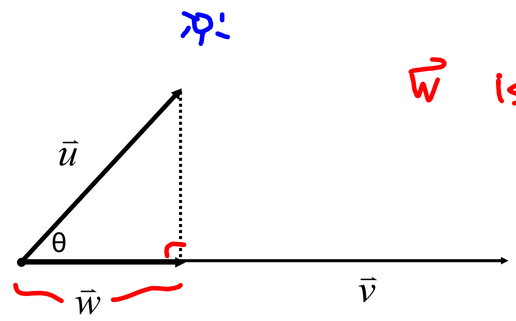
$$\|\vec{u}\| = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{29}$$
$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{30}$$

$$25 = \sqrt{29}\sqrt{30} \cos\theta$$

$$\frac{25}{\sqrt{870}} = \cos\theta$$



$$\theta = \arccos\left(\frac{25}{\sqrt{870}}\right) \simeq 32.05°$$



$\vec{w}$ is in same direction as $\vec{v}$

$\Rightarrow \vec{w} = c\vec{v}$ ①

(c is a scalar)

Also from trigonometry, we see

$$\cos\theta = \frac{\|\vec{w}\|}{\|\vec{u}\|} \Rightarrow \|\vec{w}\| = \|\vec{u}\|\cos\theta \quad ②$$

① $\|\vec{w}\| = |c|\,\|\vec{v}\|$

equate ① and ②

$$|c|\,\|\vec{v}\| = \|\vec{u}\|\cos\theta$$

$$\Rightarrow c = \frac{\|\vec{u}\|}{\|\vec{v}\|}\cos\theta$$

(assume $|c| = c$)

$$\Rightarrow \vec{w} = \left(\frac{\|\vec{u}\|}{\|\vec{v}\|}\cos\theta\right)\vec{v}$$

but remember $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$

$$\Rightarrow \vec{w} = \left(\frac{\|\vec{u}\|\,\|\vec{v}\|\cos\theta}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

$$\boxed{\vec{w} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \text{pr}_{\vec{v}}\,\vec{u}}$$

"projection of $\vec{u}$ onto $\vec{v}$"

$$\vec{w} = \left(\vec{u}\cdot\frac{\vec{v}}{\|\vec{v}\|}\right)\left(\frac{\vec{v}}{\|\vec{v}\|}\right) = (\vec{u}\cdot\hat{v})\,\hat{v}$$

EX 4 Let $\vec{u} = \langle 1,6,-2 \rangle$ and $\vec{v} = \langle -3,2,5 \rangle$. Find the vector projection of $\vec{u}$ onto $\vec{v}$.

$$pr_{\vec{v}}\vec{u} = (\vec{u} \cdot \hat{v})\,\hat{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle -3,2,5 \rangle}{\sqrt{9+4+25}} = \left\langle \frac{-3}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$$

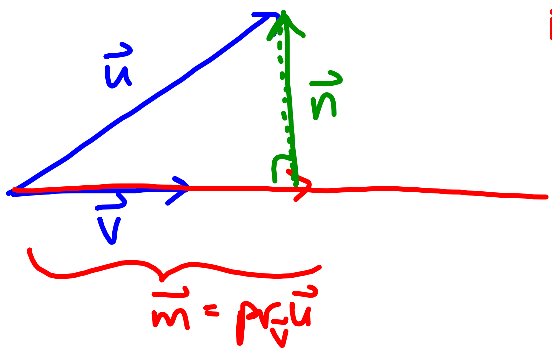
$$pr_{\vec{v}}\vec{u} = \left(\langle 1,6,-2 \rangle \cdot \left\langle \frac{-3}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle\right)\hat{v}$$

$$= \left(\frac{1}{\sqrt{38}}\right)\left(1(-3)+6(2)+-2(5)\right)\hat{v}$$

$$= \frac{-1}{\sqrt{38}} \left\langle \frac{-3}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$$

$$\boxed{pr_{\vec{v}}\vec{u} = \left\langle \frac{3}{38}, \frac{-2}{38}, \frac{-5}{38} \right\rangle}$$

EX 5 If $\vec{u} = e\hat{i} + \pi\hat{j} + \hat{k}$ and $\vec{v} = \langle 1,1,0 \rangle$, express $\vec{u}$ as the sum of vectors $\vec{m}$ and $\vec{n}$, such that $\vec{m} \parallel \vec{v}$ and $\vec{n} \perp \vec{v}$.



$$\vec{m} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}$$

$$= \frac{\langle e, \pi, 1 \rangle \cdot \langle 1,1,0 \rangle}{1^2 + 1^2 + 0^2} \vec{v}$$

$$\vec{m} = \left(\frac{e+\pi}{2} \right) \vec{v}$$

$$\vec{m} = \left(\frac{e+\pi}{2} \right) \langle 1,1,0 \rangle$$

$$\boxed{\vec{m} = \left\langle \frac{e+\pi}{2}, \frac{e+\pi}{2}, 0 \right\rangle}$$

Notice: $\vec{u} = \vec{m} + \vec{n}$

if $\vec{u} = \vec{m} + \vec{n}$, then

$$\vec{n} = \vec{u} - \vec{m}$$

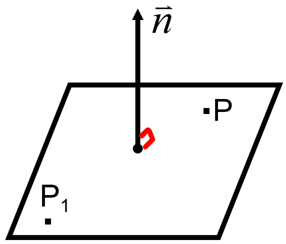
$$\vec{n} = \langle e, \pi, 1 \rangle - \left\langle \frac{e+\pi}{2}, \frac{e+\pi}{2}, 0 \right\rangle$$

$$\vec{n} = \left\langle e - \frac{e+\pi}{2}, \pi - \frac{e+\pi}{2}, 1 \right\rangle$$

$$\boxed{\vec{n} = \left\langle \frac{e-\pi}{2}, \frac{\pi-e}{2}, 1 \right\rangle}$$

Planes

Given a plane with normal vector $\vec{n} = \langle A, B, C \rangle$ and a point $P_1(x_1, y_1, z_1)$ in the plane, every other point $P(x,y,z)$ in the plane will satisfy $\boxed{\overrightarrow{P_1P} \cdot \vec{n} = 0}$.



(normal vector orthogonal to every vector in plane)

$\overrightarrow{P_1P} = \langle x - x_1, y - y_1, z - z_1 \rangle$

$\vec{n} = \langle A, B, C \rangle$

$\overrightarrow{P_1P} \cdot \vec{n} = \langle x - x_1, y - y_1, z - z_1 \rangle \cdot \langle A, B, C \rangle = 0$

$$\boxed{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0}$$

EX 6 Find the equation of a plane that goes through the origin with normal vector $\vec{n} = \langle 1,2,3 \rangle$.

A B C

$P_1 = (0,0,0)$

$\overrightarrow{P_1P} = \langle x - 0, y - 0, z - 0 \rangle = \langle x, y, z \rangle$

$\langle x, y, z \rangle \cdot \langle 1, 2, 3 \rangle = 0$

$$\boxed{x + 2y + 3z = 0}$$

Note: If I just gave you this plane eqn, we immediately know $\langle 1,2,3 \rangle$ is normal vector.

EX 7 Find the equation of the plane through $\overset{P_1}{(1,-3,4)}$ perpendicular to

x_1, y_1, z_1

$\vec{n} = \langle -1, 2, -1 \rangle$.

A B C

$\overrightarrow{P_1P} = \langle x-1, y+3, z-4 \rangle$

$\overrightarrow{P_1P} \cdot \vec{n} = 0$

$\langle x-1, y+3, z-4 \rangle \cdot \langle -1, 2, -1 \rangle = 0$

$-(x-1) + 2(y+3) - (z-4) = 0$

$-x + 2y - z + 11 = 0$

$\boxed{11 = x - 2y + z}$

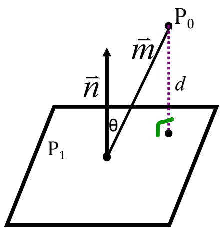
Note:

For any given plane, the most important feature of the normal vector is the direction.

Therefore, we can use any scaled version of the normal vector when determining the equation of a plane.

(i.e. normal vector is not unique)

Distance from a point $P_0(x_0, y_0, z_0)$ to a plane $Ax+By+Cz=D$

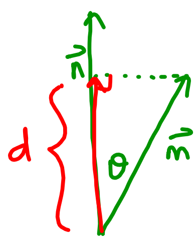


(P_0 not on plane)

$$\boxed{d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2+B^2+C^2}}}$$

notice $\vec{m}$ = vector from P_1 to P_0

$\vec{m} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle$



from trigonometry,

$d = \big|\, \|\vec{m}\| \cos\theta \,\big|$ (abs. value to make sure it's positive length)

$\Rightarrow d = \dfrac{\|\vec{m}\|\,\|\vec{n}\|\,|\cos\theta|}{\|\vec{n}\|}$

$= \dfrac{|\vec{m}\cdot\vec{n}|}{\|\vec{n}\|}$

$$\Rightarrow d = \frac{|\langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle \cdot \langle A, B, C\rangle|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{\sqrt{A^2+B^2+C^2}}$$

$$d = \frac{|Ax_0 + By_0 + Cz_0 - (Ax_1 + By_1 + Cz_1)|}{\sqrt{A^2+B^2+C^2}}$$

but P_1 is on the plane

$\Rightarrow Ax_1 + By_1 + Cz_1 = D$

$$\Rightarrow d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2+B^2+C^2}}$$

EX 8 Find the distance between the parallel planes

(1) $-3x + 2y + z = 9$ and (2) $\overset{A}{6}x - \overset{B}{4}y - \overset{C}{2}z = \overset{D}{19}$.

find a pt on plane (1), then find the distance from that pt to plane (2)

on plane (1), $P_0(\underset{x_0}{0}, \underset{y_0}{0}, \underset{z_0}{9})$

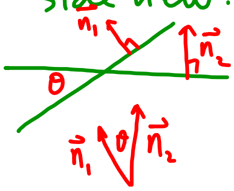
$\vec{n} = \langle \underset{A}{6}, \underset{B}{-4}, \underset{C}{-2} \rangle$

$$d = \frac{|6(0) + -4(0) + -2(9) - 19|}{\sqrt{36+16+4}} = \frac{37}{\sqrt{56}} = \frac{37}{2\sqrt{14}}$$

EX 9 Find the (smaller) angle between the two planes,

(1) $-3x + 2y + 5z = 7$ and (2) $4x - 2y - 3z = 2$.

side view:



note: this is equivalent to finding angle between the 2 normal vectors

$\vec{n}_1 = \langle -3, 2, 5 \rangle$, $\vec{n}_2 = \langle 4, -2, -3 \rangle$

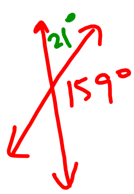
know $\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\| \|\vec{n}_2\| \cos\theta$

$$\langle -3, 2, 5 \rangle \cdot \langle 4, -2, -3 \rangle = \left(\sqrt{9+4+25}\right)\left(\sqrt{16+4+9}\right)\cos\theta$$

$$-12 - 4 - 15 = \sqrt{38}\sqrt{29}\cos\theta$$

$$\cos\theta = \frac{-31}{\sqrt{38}\sqrt{29}}$$

$$\theta = \cos^{-1}\left(\frac{-31}{\sqrt{38(29)}}\right) \approx 159^\circ$$



$\theta = 21^\circ$ smaller angle between $\vec{n}_1$ & $\vec{n}_2$

(the angle between the planes)