

# Math 2210 #31

## Gauss's Divergence Theorem

### Gauss's Divergence Theorem

Let  $\vec{F}(x, y, z)$  be a vector field continuously differentiable in the solid,  $S$ .

$S$  a 3-D solid

$\hat{n}$  unit outer normal to the surface  $\partial S$

$\text{div}\vec{F}$  divergence  $\vec{F}$

Then

$$\int_{\partial S} \vec{F}(x, y, z) \cdot \hat{n} \, dS = \iiint_D \text{div}\vec{F} \, dV$$

The rate of flow through a boundary of  $S = \iint_{\partial S} \vec{F}(x, y, z) \cdot \hat{n} dS$

If there is net flow out of the closed surface, the integral is positive.

If there is net flow into the closed surface, the integral is negative.

This integral is called "flux of  $\vec{F}$  across a surface  $\partial S$ ".  $\vec{F}$  can be any vector field, not necessarily a velocity field

Gauss's Divergence Theorem tells us that the flux of  $\vec{F}$  across  $\partial S$  can be found by integrating the divergence of  $\vec{F}$  over the region enclosed by  $\partial S$ .

### **EX 1**

$\hat{F}(x, y, z) = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$   
S is the hemisphere  $0 < z < \sqrt{a^2 - x^2 - y^2}$   
Calculate  $\iint_{\partial S} \vec{F} \cdot \vec{n} dS$ .

**EX 2**

$$\hat{F}(x, y, z) = 27\hat{i} + x\hat{j} + z^2\hat{k}$$

$S$  is the solid cylindrical shell  $1 \leq x^2 + y^2 \leq 4, 0 \leq z \leq 2$

Calculate  $\iint_{\partial S} \vec{F} \cdot \vec{n} dS$ .

**EX 3**

$$\hat{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$$

$S$  is the solid enclosed by  $x + y + z = 1, x = 0, y = 0, z = 0$

Calculate  $\iint_{\partial S} \vec{F} \cdot \vec{n} dS$ .

## EX 4

Define  $\vec{E}(x, y, z)$  to be the electric field created by a point-charge,  $q$  located at the origin.

$$\vec{E}(x, y, z) = q \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

Find the outward flux of this field across a sphere of radius  $a$  centered at the origin.

We cannot apply the divergence theorem to a sphere of radius  $a$  around the origin because our vector field is NOT continuous at the origin.

Applying it to a region between two spheres, we see that

$$\text{Flux} = \iiint_{\text{annular region}} \text{div}\vec{E} dV = 0$$

because  $\text{div}\vec{E} = 0$ .

The field entering from the sphere of radius  $a$  is all leaving from sphere  $b$ , so

$$\iint_{\substack{\text{sphere} \\ r=a}} \vec{F} \cdot \vec{n} dS = \iint_{\substack{\text{sphere} \\ r=b}} \vec{F} \cdot \vec{n} dS$$

To find flux: directly evaluate

$$\iint_{\text{sphere}} \vec{F} \cdot \hat{n} dS = \iint_{\text{sphere}} q \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{3/2}} \cdot \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{\sqrt{(x^2 + y^2 + z^2)}} dS$$