

Math 2210 #29

Green's Theorem

Goal:

Describe the relation between the way a fluid flows along or across the boundary of a plane region and the way fluid moves around inside the region.

Circulation or flow integral

Assume $\vec{F}(x, y)$ is the velocity vector field of a fluid flow. At each point (x, y) on the plane, $\vec{F}(x, y)$ is a vector that tells how fast and in what direction the fluid is moving at the point (x, y) .

Assume $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $t \in [a, b]$, is parameterization of a closed curve lying in the region of fluid flow.

Let $\vec{F}(x, y) = M(x, y)\hat{i} + N(x, y)\hat{j}$.

We want to measure "how much" fluid is moving along the curve $\vec{r}(t)$.

EX 1

Let $\vec{r}(t)$ be the parameterization of the unit circle centered at the origin. Draw these vector fields and think about how the fluid moves around that circle.

$$\vec{F}(x, y) = -2\hat{i}$$
$$\vec{F}(x, y) = \frac{-y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}}$$

When $\vec{F}(x, y)$ is parallel to the tangent line at a point, then the maximum flow is along a circle.

When $\vec{F}(x, y)$ is perpendicular to the tangent line at a point, then there is no flow along the circle.

So $\vec{F}(x, y) \cdot \vec{T}(x, y)$ measures the flow along the circle where $\vec{T}(x, y) = \vec{r}'(t)$.

We define the circulation of \vec{F} along C , a parameterized curve from $t = a$ to $t = b$ as

$$\int_a^b \vec{F}(x, y) \cdot \vec{r}'(t) dt = \int_a^b \vec{F} \cdot d\vec{r} = \int_{t=a}^{t=b} M dx + N dy$$

EX 2

Given $C: x = a \cos t, t \in [0, 2\pi]$

$$y = a \sin t,$$

find the circulation along C for each of these.

2a)

$$\vec{F}_1(x, y) = 2\hat{i}$$

2b)

$$\vec{F}_2(x, y) = \frac{-y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}}$$

Flux across a curve

Given $\vec{F}(x, y) = M\hat{i} + N\hat{j}$ (vector velocity field) and a curve C , with the parameterization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $t \in [a, b]$ such that C is a positively oriented, simple, closed curve.

We want to know the rate at which a fluid is entering and leaving the area of the region enclosed by a curve, C . This is called flux.

$\vec{F}(x, y) \cdot \vec{n}(x, y)$ is the component of \vec{F} perpendicular to the curve, so flux = $\oint_C \vec{F} \cdot \vec{n} ds$.

Now to find

$$\begin{aligned}\vec{n} &= \vec{T} \times \hat{k} \\ &= \left(\frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} \right) \times \hat{k} \\ &= \frac{dy}{ds} \hat{i} - \frac{dx}{ds} \hat{j}\end{aligned}$$

This means

$$\begin{aligned}\vec{F} \cdot \vec{n} &= M \frac{dy}{ds} - N \frac{dx}{ds} \\ \text{flux} &= \oint_C \left(M \frac{dy}{ds} - N \frac{dx}{ds} \right) ds \\ &= \oint_C M dy - N dx\end{aligned}$$

EX 3

Find the flux across $C: \vec{r}(t) = (a \cos t)\hat{i} + (a \sin t)\hat{j}$, $t \in [0, 2\pi]$

3a)

$$\vec{F}_1(x, y) = -2\hat{i}$$

3b)

$$\vec{F}_2(x, y) = \frac{-y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

3c)

$$\vec{F}_3(x, y) = x\hat{i} + y\hat{j} = (a \cos t)\hat{i} + (a \sin t)\hat{j}$$

Two Forms of Green's Theorem in The Plane

First form

Let $\vec{F}(x, y) = M\hat{i} + N\hat{j}$

Let C be a simple, closed, positively oriented curve enclosing a region R in the xy -plane.

$$\oint_C Mdy - Ndx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \nabla \cdot \vec{F} dA$$

Second form

Let $\vec{F}(x, y) = M\hat{i} + N\hat{j}$

Let C be a simple, closed, positively oriented curve enclosing a region R in the xy -plane.

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \vec{T} ds = \iint_R \nabla \times \vec{F} \cdot \hat{k} dA$$

EX 5

Verify both forms of Green's theorem for the field

$$\vec{F}(x, y) = (x - y)\hat{i} + x\hat{j}$$

and the region R bounded by the circle

$$C: \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, t \in [0, 2\pi].$$

EX 6

Evaluate the integral $\oint (xydy - y^2 dx)$ where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

EX 7

Calculate the flux of the field $\vec{F}(x, y) = x\hat{i} + y\hat{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.