Math 2210 #27
Line Integrals

Let's review parameterization of curves.

The length of a parameterized curve in 2-D $(x(t), y(t)), t \in [a, b]$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

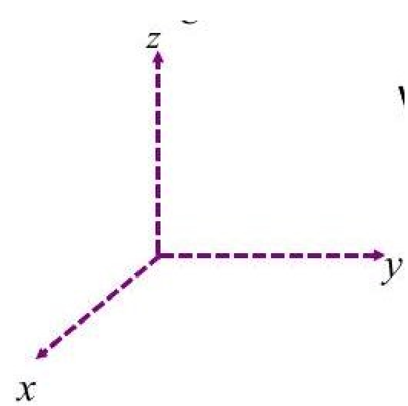
In 3-D if $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$,
then the length of a curve is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Suppose $f(x, y)$ is a function whose domain contains the curve

$$C: \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}, t \in [a, b].$$

The line integral of f along the curve C from a to b is defined as $\int_C f(x, y)ds$, where ds = arc length differential.



We know that $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Line integral = $\int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_C f(x, y) ds$

In 3 variables

$$\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t))|\vec{v}(t)|dt$$
$$= \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

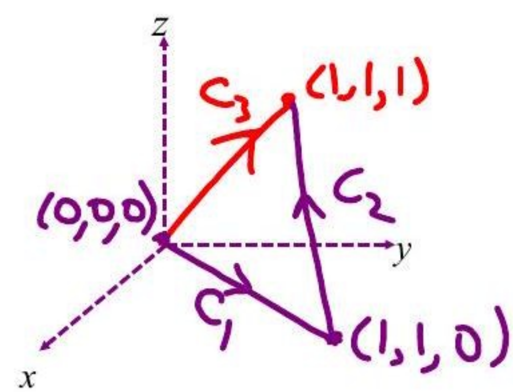
where

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$
$$\vec{v}(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}$$

EX 1

The figure shows two different paths, $C_1 \cup C_2$ and C_3.

Find $\int_{C_3} (x - 3y^2 + z)ds$ and $\int_{C_1 \cup C_2} (x - 3y^2 + z)ds$.



EX 2

A thin wire is bent in the shape of the semicircle

$$x = a\cos\ t, t \in [0, \pi], a > 0$$
$$y = a\sin\ t$$

If the density of the wire is proportional to the distance from the x-axis, find the mass of the wire.

Work

The goal is to calculate the work done by a vector field $\overrightarrow{F(x,y,z)}$ in moving an object along a curve C with parameterization.

$$C: \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, t \in [a, b]$$

The work done to move the object at (x, y, z) by a small vector, $\Delta\vec{r}$ is

$$\Delta W = \vec{F}(x,y,z) \cdot \Delta'\vec{r}(x,y,z)$$
$$W = \int_C \vec{F} \cdot d\vec{r}$$

Formula for calculating work

If $\vec{F} = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$
$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$$

where

$$M = M(x,y,z)$$
$$N = N(x,y,z)$$
$$P = P(x,y,z)$$

then $W = \int_C \vec{F} \cdot d\vec{r} =$

EX 3

Find the work done by an inverse square law force field

$$\vec{F}(x,y,z) = \frac{-c\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)}{\sqrt{x^2+y^2+z^2}}$$

in moving a particle along the straight line curve
from (0,3,0) to (4,3,0).
Note: If $c > 0$, then the work done is negative.

EX 4

Evaluate $\int_C (2x+9z)ds$, where C is the curve given by

$$x = t, y = t^2, z = t^3, t \in [0,1]$$

EX 5

Evaluate $\int_C (ydx + x^2dy)$, where C is the curve given by

$$x = 2t, y = t^2 - 1, t \in [0,2]$$