Math 2210 #26
Vector Fields

A Vector Field on a domain in space or in the plane is a function that assigns a vector to each point in the space.

EX 1

1a)

Attach a projectile's velocity vector to each point of its trajectory.

Domain: trajectory

Range: velocity field

1b)

Attach the gradient vector of a function to each point in the function's domain.

1c)

Attach a velocity vector to each point of a 3-D fluid flow.

EX 2

Plot the vector fields for each of these vector functions.

2a)

$\vec{F}(x,y) = x\hat{\imath} + y\hat{\jmath}$

2b)

$\vec{F}(x,y) = \frac{x\hat{\imath}+y\hat{\jmath}}{\sqrt{x^2+y^2}}$

2c)

$\vec{F}(x,y) = \frac{-y\hat{\imath}+x\hat{\jmath}}{\sqrt{x^2+y^2}}$

2d)

$\vec{F}(x,y) = \frac{-(x\hat{\imath}+y\hat{\jmath})}{(x^2+y^2)^{3/2}}$

Scalar field $$\text{f}: \mathbb{R}^3 \rightarrow \mathbb{R}$$ $$(x, y, z) \rightarrow f(x, y, z)$$	Vector field $$\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$ $$(x, y, z) \rightarrow \langle M, N, P \rangle$$ where $M = M(x,y,z)$, $N = N(x,y,z)$, $P = P(x,y,z)$
Gradient of scalar field $$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$	Divergence of vector field $$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$ (scalar)

Curl of vector field

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$
$$= \hat{\imath}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Think of ∇ as a vector-valued operator.

Then $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$

$$\text{div}\vec{F} = \nabla \cdot \vec{F}$$
$$= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(M\hat{\imath} + N\hat{\jmath} + P\hat{k}\right)$$
$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$
$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Note: If $\vec{F}$ is a velocity field for a fluid, then div$\vec{F}$ measures the tendency to diverge away from/toward a point.

div$\vec{F} > 0$ away
div$\vec{F} < 0$ toward

curl$\vec{F}$ - the direction about which the fluid rotates most rapidly.

$\|$ curl $\vec{F}\|$ = speed of this rotation

EX 3

Let $\vec{F}(x, y, z) = e^x \cos y\, \hat{\imath} + e^x \sin y\, \hat{\jmath} + z\hat{k}$

find $\nabla \cdot \vec{F}$
$\nabla \times \vec{F}$

EX 4

Show that

4a)

$\nabla \cdot \left(\nabla \times \vec{F}\right) = 0$ for any $\vec{F}(x, y, z)$

4b)

$\nabla \times (\nabla f) = \vec{0}$ for any $f(x, y, z)$

A vector field is called conservative if

$$\vec{F}(x, y, z) = \nabla f(x, y, z) \text{ for some } f\colon \mathbb{R}^3 \to \mathbb{R}$$

then f is called the potential function.

EX 5

Let $\vec{F}(x, y, z) = \frac{-c\vec{r}}{\|\vec{r}\|^3}$

$$f(x, y, z) = \frac{c}{\sqrt{x^2 + y^2 + z^2}}$$

show that

$$\vec{F} = \nabla f$$