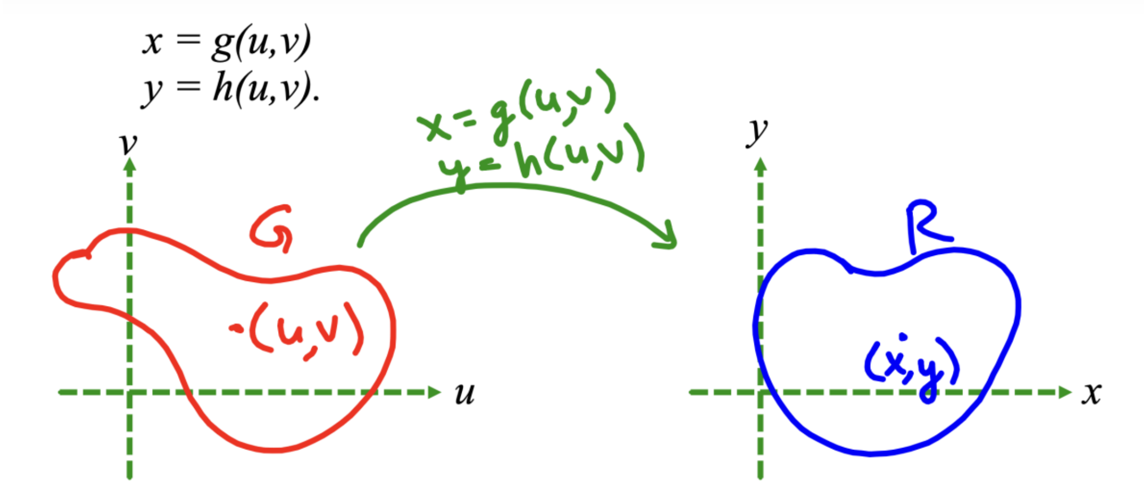
Math 2210 #25
Change of Variables
(Jacobian Method)

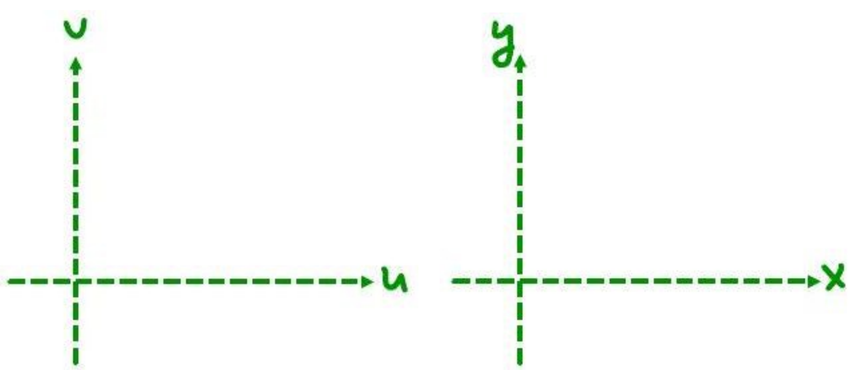
Transformations from a region G in the uv-plane to the region R in the xy-plane are done by equations of the form

$$x = g(u, v)$$
$$y = h(u, v).$$



EX 1

$x = \frac{u+v}{2}, y = \frac{u-v}{2}$, G is the rectangle given by $\begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 1. \end{array}$



How is the integral of $f(x, y)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G?

$$\iint_R f(x, y)dxdy = \iint_G f(g(u, v), h(u, v))|J(u, v)|dudv$$

where $J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

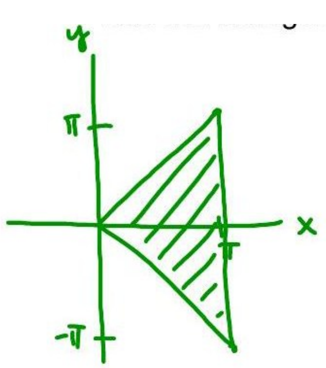
EX 2

For polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, what is $J(r, \theta)$?

$$J(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

EX 3

Evaluate $\iint_R \cos(x - y)\sin(x + y)\, dA$ where R is the triangle in the xy-plane with vertices at (0,0), (π, −π) and (π, pi). Use the change of variables $u = x - y$ and $v = x + y$.

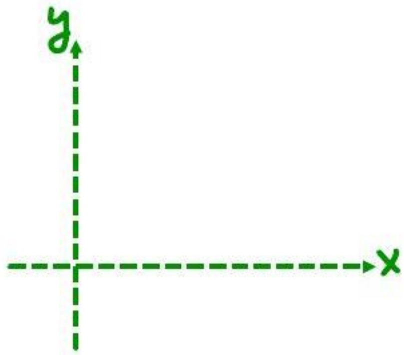


EX 4

Evaluate $\iint_R 5(x^2+y^2)dxdy$ where R is the region in Q_I bounded by

$x^2+y^2=9, x^2+y^2=16, y^2-x^2=1, y^2-x^2=9.$

Hint: Use $u=x^2+y^2$ and $v=y^2-x^2$ to transform R into a much nicer region (G).



Change of variables in 3 dimensions.
If $x = g(u, v, w)$
$y = h(u, v, w)$
$z = j(u, v, w)$
then

$$\iiint_{\underline{R}} f(x, y, z)dxdydz = \iiint_{G} f\big(g(u, v, w), h(u, v, w)j(u, v, w)\big)|J(u, v, w)|dudvdw$$

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

EX 5

Let's check the Jacobian for spherical coordinates.

$$x = \rho \cos\theta \sin\varphi$$
$$y = \rho \sin\theta \sin\varphi$$
$$z = \rho \cos\varphi$$

$$J(\rho, \theta, \varphi) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$