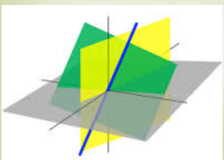
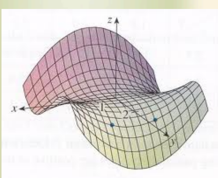


$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

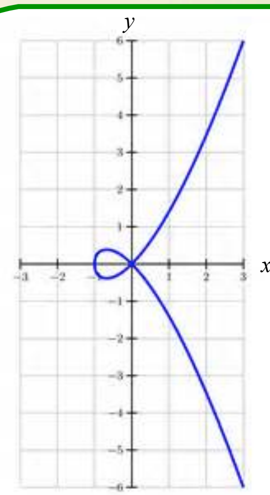


$$\int_0^1\int_0^{2y} xy\,dx\,dy = \int_0^1\left[\frac{x^2}{2}y\right]_{x=0}^{x=2y}dy$$

$$= \int_0^1\frac{(2y)^2}{2}y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Parametric Representations of Plane Curves



$x = t^2 - 1$

$y = t^3 - t$

A plane curve is a 2-dimensional curve given by

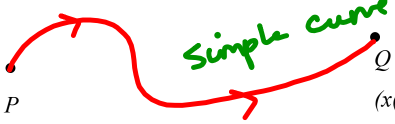
$$x = f(t) \qquad y = g(t) \qquad t \in I$$

(curve is given parametrically)

where f and g are continuous functions on the interval I, $[a,b]$.

t is the parameter. $\quad t \in [a,b]$

simple curve, not closed



P

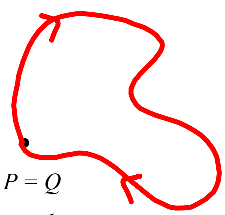
$(x(a), y(a))$

initial endpoint

Q

$(x(b), y(b))$

terminal endpoint

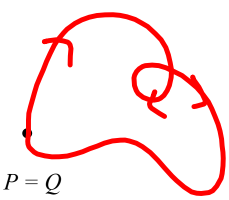


$P = Q$

simple,

closed

curve

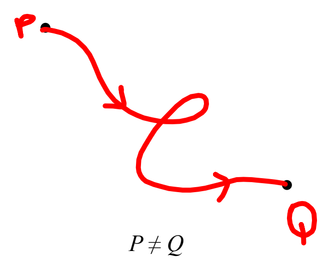


$P = Q$

not simple,

closed

curve



$P \neq Q$

not simple,

not closed

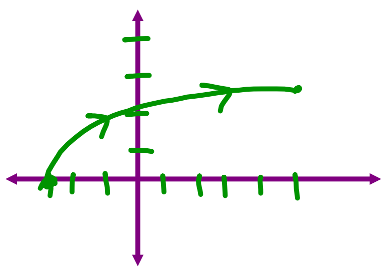
curve

(simple: non-self-intersecting)

It can be hard to recognize the shape of a curve when given parametically. Sometimes it is possible to eliminate the parameter.

EX 1 Eliminate the parameter and sketch this curve.

① $x = t - 3$ ② $y = \sqrt{t}$ $0 \le t \le 8$



① $t = x + 3$

② $y = \sqrt{x+3}$ (know the graph)

start $t = 0$, $x = 0 - 3 = -3$, $y = 0$

end $t = 8$, $x = 8 - 3 = 5$, $y = \sqrt{8} = 2\sqrt{2}$

EX 2 Eliminate the parameter t, graph the curve and tell if it is simple and closed.

① $x=\sqrt{t-3}$ ② $y=\sqrt{4-t}$ $\underline{3 \le t \le 4}$

① $x^2 = t-3$

$t = x^2+3$

② $y=\sqrt{4-(x^2+3)}$

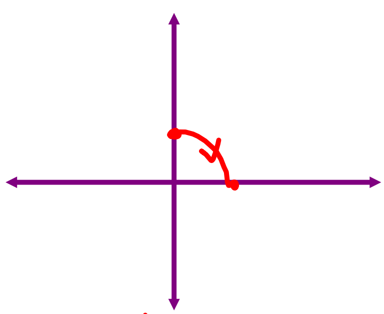
$y=\sqrt{1-x^2}$

$y^2 = 1-x^2$

$\underline{x^2+y^2=1}$ (basic shape) (circle)

start: $t=3$, $x=0$, $y=1$

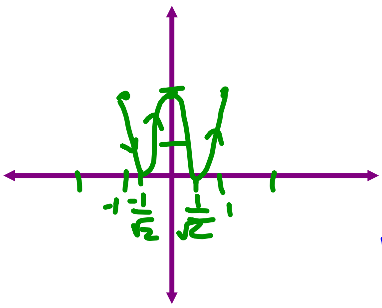
end: $t=4$, $x=1$, $y=0$



- yes, it's simple
- no, it's not closed

EX 3 Eliminate the parameter θ, graph the curve and tell if it is simple and closed.

① $x = \sin\theta$ ② $y = 2\cos^2(2\theta)$ $\theta \in \Re$



②

$y = 2(\cos(2\theta))^2$

$y = 2(1 - 2\sin^2\theta)^2$

$y = 2(1 - 4\sin^2\theta + 4\sin^4\theta)$

use trig identity: (double angle)

$\cos(2\theta) = 1 - 2\sin^2\theta$

plug in $x = \sin\theta$:

$y = 2(1 - 4x^2 + 4x^4)$

$y = 8x^4 - 8x^2 + 2$

$\theta \in \mathbb{R}$

① $x = \sin\theta \Rightarrow x \in [-1, 1]$

② $y = 2\cos^2(2\theta) \Rightarrow y \in [0, 2]$

$0 \le \cos^2(2\theta) \le 1$

$0 \le 2\cos^2(2\theta) \le 2$

$y = 2(1 - 2x^2)^2$

zeros: when $2x^2 = 1$

$x = \pm 1/\sqrt{2}$

pts $(-1, 2)$ $(0, 2)$

$(1, 2)$ $(\pm 1/\sqrt{2}, 0)$

Theorem A

Let f and g be continuously differentiable with $f'(t) \neq 0$ on $t \in (\alpha, \beta)$.

Then the parametric equations $x = f(t)$ and $y = g(t)$

define y as a differentiable function of x and

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} \quad where\ y' = \frac{dy}{dx}}$$

EX 4 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (without eliminating the parameter.)

a) $x = \sqrt{3}\theta^2 \qquad y = -\sqrt{3}\theta^3 \qquad \theta \neq 0$

$$\frac{dy}{d\theta} = -3\sqrt{3}\,\theta^2$$

$$\frac{dx}{d\theta} = 2\sqrt{3}\,\theta$$

$$\frac{dy'}{d\theta} = -3/2$$

$$y' = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3\cancel{\sqrt{3}}\,\theta^2}{2\cancel{\sqrt{3}}\,\theta} = \frac{-3}{2}\theta$$

$$\frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta} = \frac{-3/2}{2\sqrt{3}\,\theta} = \frac{-3}{4\sqrt{3}\,\theta} = \frac{-\sqrt{3}}{4\theta}$$

b) $x = \dfrac{2}{1+t^2} \qquad y = \dfrac{2}{t(1+t^2)} \qquad t \neq 0$

$$\frac{dx}{dt} = \frac{-2(2t)}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{0-2(1+3t^2)}{t^2(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{-2(1+3t^2)}{t^2(1+t^2)^2}$$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-2(1+3t^2)}{t^2\cancel{(1+t^2)^2}}}{\frac{-4t}{\cancel{(1+t^2)^2}}} = \frac{-2(1+3t^2)}{t^2(-4t)} = \boxed{\frac{1+3t^2}{2t^3}}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1+3t^2}{2t^3}\right) = \frac{d}{dt}\left(\frac{1}{2}t^{-3} + \frac{3}{2}t^{-1}\right)$$

$$= \frac{-3}{2}t^{-4} - \frac{3}{2}t^{-2} = \frac{-3}{2}\left(\frac{1}{t^4} + \frac{1}{t^2}\right)$$

$$= \frac{-3}{2}\left(\frac{t^2+1}{t^4}\right)$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\frac{-3}{2}\left(\frac{t^2+1}{t^4}\right)}{\frac{-4t}{(1+t^2)^2}}$$

$$= \frac{-3}{2}\left(\frac{t^2+1}{t^4}\right) \cdot \frac{(1+t^2)^2}{-4t}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{3(1+t^2)^3}{8t^5}}$$

Length of a curve: $L = \int_a^b \underbrace{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt}_{ds}$

(arc length)

ds = "a little bit of arc length"

EX 5 Find the length of the curve given by

$$x = \sin\theta - \theta\cos\theta \qquad \theta \in [\pi/4, \pi/2]$$
$$y = \cos\theta + \theta\sin\theta$$

$$L = \int_{\pi/4}^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = \cos\theta - (\cos\theta + \theta(-\sin\theta))$$
$$= \cos\theta - \cos\theta + \theta\sin\theta$$
$$= \theta\sin\theta$$

$$\frac{dy}{d\theta} = -\cancel{\sin\theta} + (\cancel{\sin\theta} + \theta\cos\theta)$$
$$= \theta\cos\theta$$

note: $\sqrt{\theta^2} = |\theta|$

$$L = \int_{\pi/4}^{\pi/2} \sqrt{(\theta\sin\theta)^2 + (\theta\cos\theta)^2}\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \sqrt{\theta^2 \underbrace{(\sin^2\theta + \cos^2(\theta))}_{=1}}\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \sqrt{\theta^2}\, d\theta = \int_{\pi/4}^{\pi/2} \theta\, d\theta$$

$$= \left.\frac{\theta^2}{2}\right|_{\pi/4}^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{3\pi^2}{32}$$