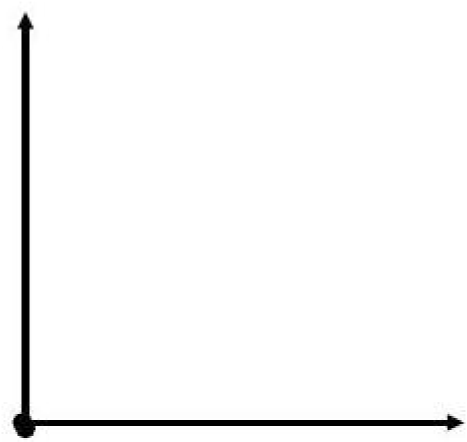
Math 2210 #18
Lagrange Multipliers

Now we will see an easier way to solve extrema problems with some constraints.

We want to optimize $f(x, y)$ subject to constraint $g(x, y) = 0$.
Graphically:

: level curves ($f(x, y) = k$)

: constraint curve



To maximize f subject to $g(x, y) = 0$ means to find the level curve of f with greatest k-value that intersects the constraint curve. It will be the place where the two curves are tangent.

Two curves have a common perpendicular line if they are tangent at that point. We know ∇f is perpendicular to its level curves. ∇g is also perpendicular to the constraint curve.

Theorem (Lagrange's Method)

To maximize or minimize $f(x, y)$ subject to constraint $g(x, y) = 0$, solve the system of equations

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \qquad g(x, y) = 0$$

for (x, y) and λ. The solutions (x, y) are critical points for the constrained extremum problem and the corresponding λ is called the Lagrange Multiplier.

Note: *Each critical point we get from these solutions is a candidate for the max/min.*

EX 1

Find the maximum value of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = 4x^2 + 9y^2 - 36 = 0.$$

EX 2

Find the least distance between the origin and the plane

$$x + 3y - 2z = 4.$$

EX 3

Find the max volume of the first-octant rectangular box (with faces parallel to coordinate planes) with one vertex at $(0,0,0)$ and the diagonally opposite vertex on the plane $3x - y + 2z = 1$.

If we have more than one constraint, additional Lagrange multipliers are used. If we want to maximize $f(x, y, z)$ subject to $g(x, y, z) = 0$ and $h(x, y, z) = 0$, then we solve

$$\nabla f = \lambda \nabla g + \mu \nabla h \text{ with } g = 0 \text{ and } h = 0$$

EX 4

Find the minimum distance from the origin to the line of intersection of the two planes.

$$x + y + z = 8 \text{ and } 2x - y + 3z = 28$$

Lagrange multipliers don't work well for constraint regions like a square or triangle because there is not one equation to represent $g(x, y) = 0$.