

Math 2210 #15

The Chain Rule

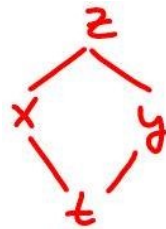
Recall: Chain rule for $y = f(g(x))$ is $y' = f'(g(x))g'(x) = \frac{df}{dg} \frac{dg}{dx}$.

Chain Rules

Theorem

Let $x = x(t)$ and $y = y(t)$ be differentiable at t and let $z = f(x, y)$ be differentiable at $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \Leftrightarrow \frac{dz}{dt} = \nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle.$$



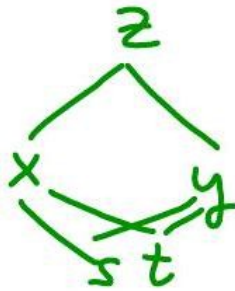
Theorem

Let $x = x(s, t)$ and $y = y(s, t)$ have first partial derivatives at (s, t) and let $z = f(x, y)$ be differentiable at $(x(s, t), y(s, t))$.

Then z has first partial derivatives given by

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$



EX 1

Find $\frac{\partial w}{\partial t}$ given $w = x^2y - y^2x, x = \cos t, y = \sin t$.
Express the answer in terms of t .

EX 2

Find $\frac{\partial w}{\partial t}$ given $w = \ln(x + y) - \ln(x - y), x = te^s, y = e^{st}$.
Express the answer in s and t .

EX 3

If $w = xy + x + y, x = r + s + t$ and $y = rst$,
find $\frac{\partial w}{\partial t} \Big|_{r=1, s=-1, t=2}$.

EX 4

Sand is pouring onto a conical pile in such a way that at a certain instant, the height is 100 inches and increasing at 3in/min. The base radius at that instant is 40 inches and increasing at 2in/min. How fast is the volume increasing at that instant?

Implicit Differentiation

Let's go back to $y = f(x)$ and assume that instead of getting y as a function of x (explicitly), we have $F(x, y) = k$ for any constant, k (i.e. y is defined implicitly). Then, we just differentiated both sides with respect to x to get dy/dx .

For example:

$$y^3 - 2xy + 3x = 4$$

Apply the same ideas:

$$xy^3 + \sec(y + z) - zx^2 = 1$$

EX 5

If $ye^{-x} + \sin(x + z) + e^z = 5$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.