Math 2210 #14
Directional Derivatives

Directional Derivatives

We know we can write

$$\frac{\partial f}{\partial x} = f_x(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(a,b) = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}$$

The partial derivatives measure the rate of change of the function at a point <u>in the direction</u> of the x-axis or y-axis. What about the rates of change in the other directions?

Definition

For any unit vector, $\hat{u} = \langle u_x, u_y \rangle$ let

$$D_{\tilde{u}} f(a,b) = \lim_{h\to 0}\frac{f(a+hu_x, b+hu_y)-f(a,b)}{h}$$

If this limit exists, this is called the directional derivative of f at the point (a, b) in the direction of $\hat{u}$.

Theorem

Let f be differentiable at the point (a,b). Then f has a directional derivative at (a, b) in the direction of $\hat{u}$. $\hat{u} = u_x\hat{\imath} + u_y\hat{\jmath}$ and $D_{\bar{u}} f(a,b) = \hat{u}\cdot\nabla f(a,b)$.

EX 1

Find the directional derivative of $f(x, y)$ at the point (a, b) in the direction of $\vec{u}$. (Note: $\vec{u}$ may not be a unit vector.)

1a)

$f(x, y) = y^2 \ln(x)$ $(a, b) = (1,4)$ $\vec{u} = \hat{\imath} - \hat{\jmath}$

1b)

$f(x, y) = 2x^2 \sin y + xy$ $(a, b) = (1, \pi/2)$ $\vec{u} = 2\hat{\imath} + \hat{\jmath}$

Maximum Rate of Change

We know $D_{\hat{u}} f(a, b) = \hat{u} \cdot \nabla f(a, b)$

$$= \|\hat{u}\| \|\nabla f(a, b)\| \cos \theta$$

What angle, θ, maximizes $D_{uf} f(a, b)$?

Theorem

The function, $z = f(x, y)$, increases most rapidly at (a, b) in the direction of the gradient (with rate $\|\nabla f(a, b)\|$) and decreases most rapidly in the opposite direction (with rate $-\|\nabla f(a, b)\|$).

EX 2

For $z = f(x,y) = x^2 + y^2$, interpret gradient vector.

EX 3

Find a vector indicating the direction of most rapid increase of $f(x,y)$ at the given point. Then find the rate of change in that direction.

3a)

$f(x,y) = e^y \sin\ x$ at $(a,b) = (5\pi/6, 0)$

3b)

$f(x,y) = x^2y - 2/(xy)$ at $(a,b) = (1,1)$

EX 4

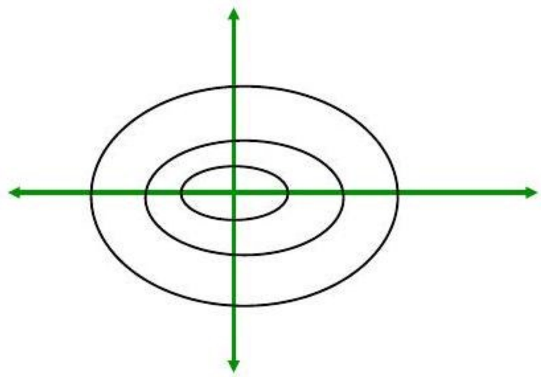
The temperature at (x,y,z) of a ball centered at the origin is $T = 100e^{-(x^2+y^2+z^2)}$.

Show that the direction of greatest decrease in temperature is always a vector pointing away from the origin.

One extra (cool) fact

Theorem

The gradient of $z = f(x, y)(w = f(x, y, z))$ at point P is perpendicular to the level curve (surface) of f through P.



EX 5

Graph gradient vectors and level curves for

$$z = f(x, y) = \frac{x^2}{9} + \frac{y^2}{25}.$$