Math 1220 #18
Infinite Sequences

Definition

An infinite sequence is an ordered arrangement of real numbers.

$$a_1, a_2, a_3, a_4, \ldots$$
$$\{a_n\}_{n=1}^{\infty}$$
$$\{a_n\}$$

iteration (explicit)	recursion (implicit)
$a_n = 5n - 3$	$a_1 = 2$ $a_n = a_{n-1} + 5 \; n \geq 2$

We can just write out the terms.

$$2,7,12,17,22, \ldots$$

Definition

Convergence

$\{a_n\}$ converges to L, written $\lim_{n\to\infty} a_n = L$
if for each positive ε there exists a corresponding positive N such that

$$n \geq N \Rightarrow |a_n - L| < \varepsilon$$

If a sequence fails to converge to a finite L, then it diverges.
Example $a_n = \frac{n}{2n-1}$

EX 1

Does $\{a_n\}$ converge? $a_n = \frac{5n^2-3n+1}{2n^2+7}$
If so, what is the limit?

Properties of limits of sequences

Assume $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, then

1. $\lim_{n\to\infty} k = k$
2. $\lim_{n\to\infty} (a_n \cdot b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, b_n \neq 0$
4. $\lim_{n\to\infty} ka_n = k\lim_{n\to\infty} a_n$
5. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$.

x is a continuous variable
n is a discrete variable

We can use l'Hopital's Rule.

EX 2

Determine if $\{a_n\}$ converges and if so, find $\lim_{n\to\infty} a_n$.

2a)

$$a_n = \frac{\ln(1/n)}{\sqrt{2n}}$$

2b)

$$a_n = \frac{n^{100}}{e^n}$$

Squeeze Theorem

If $\{a_n\}$ and $\{c_n\}$ both converge to L and
$a_n \leq b_n \leq c_n$ for $n \geq K$ (some fixed integer),
then $\{b_n\}$ also converges to L.

EX 3

Determine if $\{a_n\}$ converges and if so, find $\lim_{n\to\infty} a_n$.

$$\{a_n\} = e^{-n}\sin\, n$$

Theorem

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

EX 4

Show that if r is in this interval $(-1,1)$, then

$$\lim_{n\to\infty} r^n = 0.$$

Monotonic Sequence Theorem

If U is an upper bound for a nondecreasing sequence $\{a_n\}$, then the sequence converges to a limit A such that $A \leq U$.
Also, if L is a lower bound for a nonincreasing sequence $\{b_n\}$, then the sequence converges to a limit B such that $B \geq L$.

EX 5

Write the first four terms for this sequence. Show that it converges.

$$\{a_n\} = \frac{n}{n+1}\left(2 - \frac{1}{n^2}\right)$$