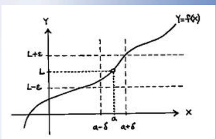
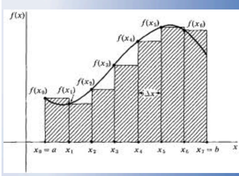


$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

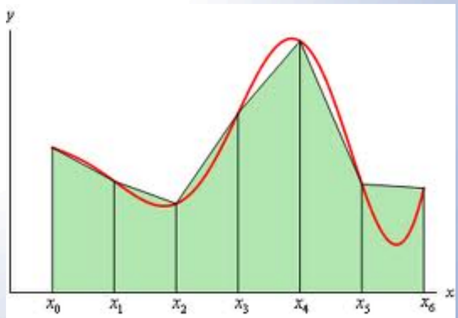
$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$



$$\lim_{max \Delta x_i \to 0} \sum_{1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

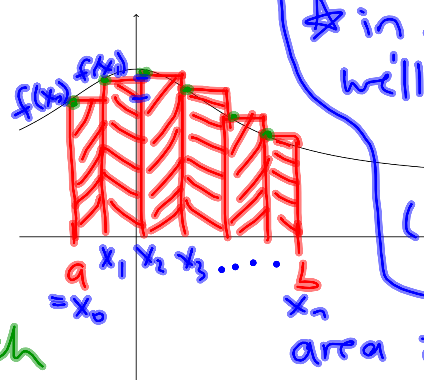
$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Numerical Integration



If f(x) is continuous, we are guaranteed that $\int_a^b f(x)\,dx$ exists, but sometimes we cannot evaluate the integral. For these cases, we use numerical methods to approximate the definite integral (area under the curve.)

1. Left Riemann Sum



adds up areas of finitely many rectangles.

✱ in each method, we'll choose $\Delta x_i = \Delta x$ (i.e. all widths are same)

choose height of each rectangle by left y-value.

area of n^{th} rectangle $= f(x_{n-1})\Delta x$

(have n rectangles total)

$$\Delta x = \frac{b-a}{n}$$

$$x_{i-1} = a + (i-1)\Delta x$$

$i = 1, .., n$

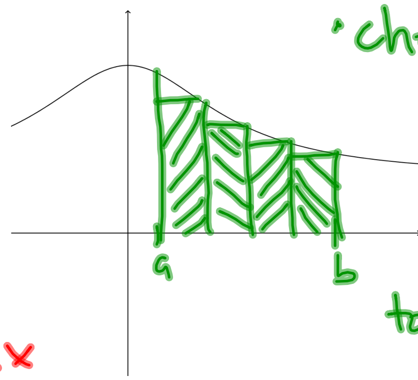
$$\Rightarrow \int_a^b f(x)\,dx \simeq \frac{b-a}{n}\sum_{i=1}^{n} f(a+(i-1)\Delta x)$$

$$= \frac{b-a}{n}\sum_{i=1}^{n} f\left(a+(i-1)\left(\frac{b-a}{n}\right)\right)$$

Error: $E_n = \frac{(b-a)^2}{2n} f'(c)$ for some $c \in [a,b]$.

2. Right Riemann Sum

area of n^{th} rectangle
$= f(x_i)\Delta x_n$



• choose the right most x values in each rectangle to compute height

let all $\Delta x_n = \Delta x$.

$\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

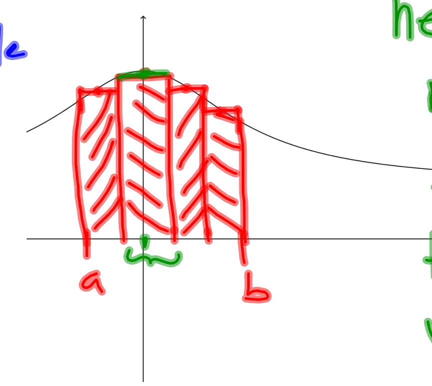
area under curve on [a,b] $\int_a^b f(x)\,dx \simeq \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\left(\frac{b-a}{n}\right)\right)$

error: $E_n = \frac{-(b-a)^2}{2n} f'(c)$ for some $c \in [a,b]$

3. <u>Midpoint Riemann Sum</u>

area of n^{th} rectangle

$= f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$



height of each rectangle is the height of the x-midpoint value for that rectangle.

$\Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x$

$x_{i-1} = a + (i-1)\Delta x$

$\Rightarrow \frac{x_i + x_{i-1}}{2} = \frac{a + i\Delta x + a + (i-1)\Delta x}{2} = \frac{2a + 2i\Delta x - \Delta x}{2}$

$= \underbrace{a + i\Delta x - \tfrac{1}{2}\Delta x}_{\text{x-value plugged into fn to get height of rectangle}}$

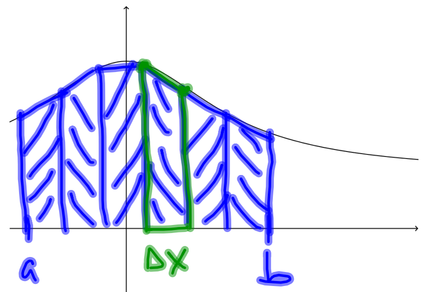
$\Rightarrow \int_a^b f(x)\,dx \simeq \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\Delta x - \tfrac{1}{2}\Delta x\right)$

$E_n = \frac{(b-a)^3}{24n^2} f''(c)$ for some c in $[a,b]$

4. Trapezoidal Rule

area of n^{th} trapezoid

$$= \frac{1}{2}\left(f(x_i)+f(x_{i-1})\right)\Delta x$$



$\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

$x_{i-1} = a + (i-1)\Delta x$

$$\int_a^b f(x)\,dx \simeq \underbrace{\frac{1}{2}\left(\frac{b-a}{n}\right)\sum_{i=1}^{n}\left[f(x_{i-1})+f(x_i)\right]}_{\text{sum of area of trapezoids}}$$

area of right trapezoid:

$$A = \overbrace{\frac{1}{2}(y_1-y_2)h}^{\text{area of }\Delta} + \underbrace{hy_2}_{\text{area of rect}}$$

$$\Rightarrow A = \frac{1}{2}(y_1+y_2)h$$

(multiply out & collect like terms)

$$\int_a^b f(x)\,dx \simeq \left(\frac{b-a}{n}\right)\left[\frac{f(a)}{2} + \underbrace{\sum_{i=1}^{n-1} f(x_i)} + \frac{f(b)}{2}\right]$$

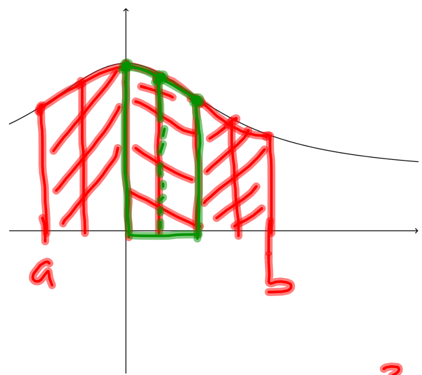
$$E_n = -\frac{(b-a)^3}{12n^2} f''(c) \quad \text{for some } c \in [a,b]$$

5. Simpson's Rule (a.k.a. Parabolic Rule)

$\Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x$

$\boxed{\text{* } n \text{ must be even}}$

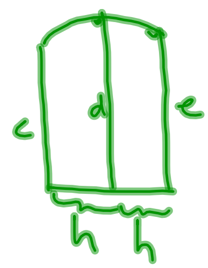


for every two "widths", we connect top 3 pts w/ parabola.

area of one parabolic piece

$$= \frac{\Delta x}{3}\left(f(x_i) + 4f(x_{i+1}) + f(x_{i+2})\right)$$

Area of parabolic piece:



$$A = \frac{h}{3}(c + 4d + e)$$

(for us, $h = \Delta x$)

c = ht (fn value) at leftmost x-value

d = ht (fn value) at x-midpt-value

$$\int_a^b f(x)\,dx \simeq \left(\frac{b-a}{n}\right)\left(\frac{1}{3}\right)\Big[\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \left(f(x_2) + 4f(x_3) + f(x_4)\right) + \left(f(x_4) + 4f(x_5) + f(x_6)\right) + \dots + \left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)\Big]$$

$$= \frac{b-a}{3n}\left[f(a) + 4\left(\sum_{i=1}^{n/2} f(a + (2i-1)\Delta x)\right) + 2\left(\sum_{i=1}^{n/2-1} f(a + 2i\Delta x)\right) + f(b)\right]$$

error: $E_n = \dfrac{-(b-a)^5}{180n^4} f^{(4)}(c)$ for some $c \in [a,b]$

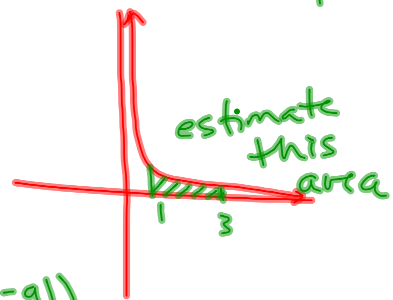
EX 1

$a=1,\ b=3$

Use methods 2, 4 and 5 to approximate this integral. $\displaystyle\int_1^3 \frac{1}{x^3}\,dx$ Let $n = 8$

$\Delta x = \frac{3-1}{8} = \frac{1}{4}$

(method 2 = right Riemann Sum,
" 4 = trapezoid rule,
" 5 = Simpson's Rule).



Right Riemann Sum Rule

2

$$\int_a^b f(x)\,dx \simeq \frac{b-a}{n}\sum_{i=1}^{n} f\Big(a+i\underbrace{\Big(\frac{b-a}{n}\Big)}_{\Delta x}\Big)$$

$f(x)=\frac{1}{x^3}$

$a=1,\ b=3,\ n=8$

$$\int_1^3 \frac{1}{x^3}\,dx \simeq \frac{1}{4}\sum_{i=1}^{8} f\Big(1+i\Big(\frac{1}{4}\Big)\Big)$$

$$=\frac{1}{4}\sum_{i=1}^{8} f\Big(\frac{4+i}{4}\Big)$$

$$=\frac{1}{4}\sum_{i=1}^{8}\left(\frac{1}{\frac{4+i}{4}}\right)^3 = \frac{1}{4}\sum_{i=1}^{8}\frac{4^3}{(4+i)^3} = 4^2\sum_{i=1}^{8}\frac{1}{(4+i)^3}$$

$$= 4^2\Big(\frac{1}{5^3}+\frac{1}{6^3}+\frac{1}{7^3}+\frac{1}{8^3}+\frac{1}{9^3}+\frac{1}{10^3}+\frac{1}{11^3}+\frac{1}{12^3}\Big)$$

$$\simeq 16(0.02119967)$$

$$\simeq 0.339199$$

$$\simeq 0.339 \simeq \int_1^3 \frac{1}{x^3}\,dx$$

$\int_1^3 \frac{1}{x^3}\,dx$ Let n = 8. Trapezoidal Rule

$\Delta x = \frac{3-1}{8} = \frac{1}{4}$

$\int_a^b f(x)\,dx \simeq \Delta x\left(\frac{f(a)}{2} + \sum_{i=1}^{n-1} f(x_i) + \frac{f(b)}{2}\right]$

$n=8,\ a=1,\ b=3,\ x_i = a + i\Delta x \qquad x_i = 1 + \frac{1}{4}i$

$\int_1^3 \frac{1}{x^3}\,dx \simeq \frac{1}{4}\left[\frac{1/1^3}{2} + \sum_{i=1}^{7} f\left(\frac{4+i}{4}\right) + \frac{1/3^3}{2}\right] \qquad x_i = \frac{4+i}{4}$

$= \frac{1}{4}\left[\frac{1}{2} + \sum_{i=1}^{7} \frac{1}{\left(\frac{4+i}{4}\right)^3} + \frac{1}{54}\right]$

$= \frac{1}{4}\left[\frac{1}{2} + \frac{1}{54} + \sum_{i=1}^{7} \frac{4^3}{(4+i)^3}\right]$

$= \frac{1}{\cancel{4}}\left(\frac{\cancel{28}\,7}{54}\right) + 4^2 \sum_{i=1}^{7} \frac{1}{(4+i)^3}$

$= \frac{7}{54} + 16\left[\frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{9^3} + \frac{1}{10^3} + \frac{1}{11^3}\right]$

$\simeq \frac{7}{54} + 16(0.02062126)$

$\simeq 0.4595698$

$\simeq 0.46 \simeq \int_1^3 \frac{1}{x^3}\,dx$

$\int_1^3 \frac{1}{x^3}dx$ Let n = 8. Simpson's Rule

$\Delta x = \frac{3-1}{8} = \frac{1}{4}$

$a + (2i-1)\Delta x$
$= 1 + (2i-1)\frac{1}{4}$
$= \frac{3}{4} + \frac{i}{2}$

$a=1,\ b=3,\ n=8$

$$\int_a^b f(x)\,dx \simeq \frac{b-a}{3n}\left[f(a) + 4\sum_{i=1}^{n/2} f(a+(2i-1)\Delta x) + 2\sum_{i=1}^{n/2-1} f(a+2i\Delta x) + f(b)\right]$$

$$\boxed{\begin{aligned} &a + 2i\Delta x \\ &= 1 + 2i\left(\tfrac{1}{4}\right) \\ &= 1 + \tfrac{1}{2}i \end{aligned}}$$

$$\int_1^3 \frac{1}{x^3}dx \simeq \frac{3-1}{3(8)}\left[\frac{1}{1^3} + 4\sum_{i=1}^{4} f\left(\frac{3}{4}+\frac{i}{2}\right) + 2\sum_{i=1}^{3} f\left(1+\frac{i}{2}\right) + \frac{1}{3^3}\right]$$

$f(x) = \frac{1}{x^3}$

$$\simeq \frac{1}{12}\left[1 + \frac{1}{27} + 4\sum_{i=1}^{4}\left(\frac{1}{\left(\frac{3+2i}{4}\right)^3}\right) + 2\sum_{i=1}^{3}\frac{1}{\left(\frac{2+i}{2}\right)^3}\right]$$

$$\boxed{\begin{aligned} &\tfrac{3}{4} + \tfrac{i}{2} \\ &= \tfrac{3}{4} + \tfrac{2i}{4} = \tfrac{3+2i}{4} \end{aligned}}$$

$$\simeq \frac{1}{12}\left[\frac{28}{27} + 4\sum_{i=1}^{4}\frac{4^3}{(3+2i)^3} + 2\sum_{i=1}^{3}\frac{2^3}{(2+i)^3}\right]$$

$$= \frac{1}{12}\left[\frac{28}{27} + 256\left(\frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3}\right) + 16\left(\frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3}\right)\right]$$

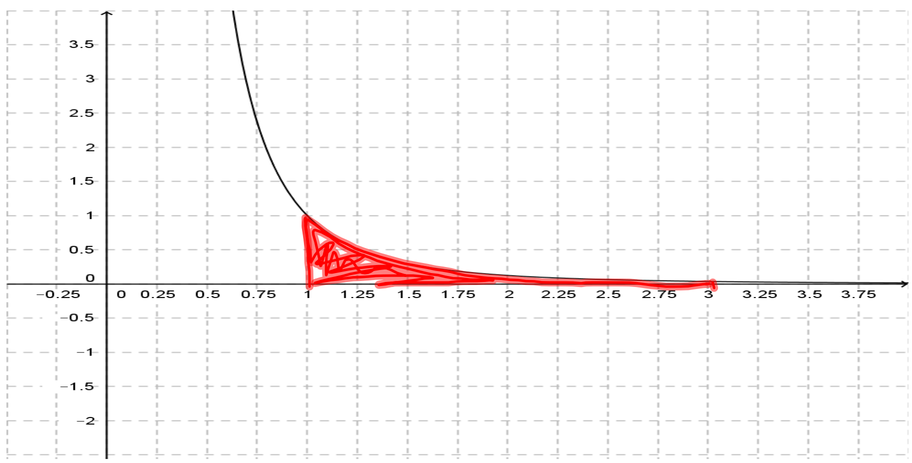
$$= \frac{7}{3(27)} + \frac{64}{3}\left(\frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3}\right) + \frac{4}{3}\left(\frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3}\right)$$

$$\simeq \frac{7}{81} + 0.27815485 + 0.0808827$$

$$\simeq 0.445457 \simeq \int_1^3 \frac{1}{x^3}dx$$

Actual Value

$$\int_1^3 \frac{1}{x^3}\,dx$$



Right Riemann Sum estimate $\simeq 0.339$

Trapezoidal Rule estimate $\simeq 0.460$

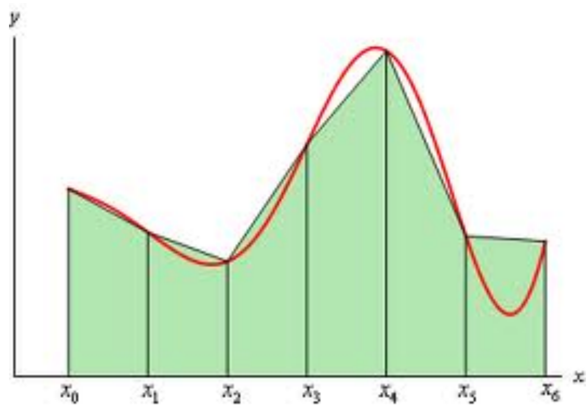
Simpson's Rule estimate $\simeq 0.445$

actual value:

$$\int_1^3 \frac{1}{x^3}\,dx = \int_1^3 x^{-3}\,dx$$

$$= \frac{x^{-2}}{-2}\Bigg|_1^3 = \frac{-1}{2x^2}\Bigg|_1^3 = \frac{-1}{2(9)} - \frac{-1}{2(1)}$$

$$= \frac{-1}{18} + \frac{1}{2} = \frac{8}{18} = \frac{4}{9} = 0.\overline{4}$$



shows trapezoidal rule