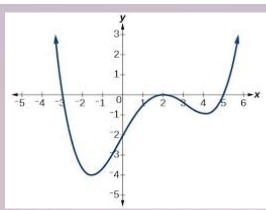

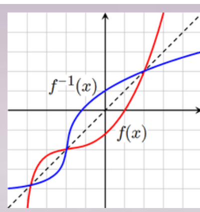

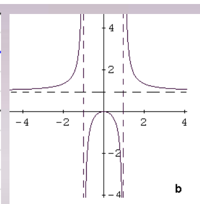

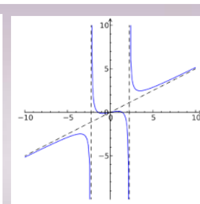

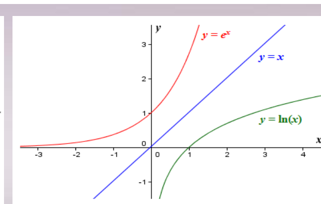

Math 1050 ~ College Algebra

27 Partial Fractions

$$-3x + 4y = 5$$
$$2x - y = -10$$
$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

Learning Objectives

- Decompose a rational expression with denominator of non-repeated linear factors into a sum of partial fractions.
- Decompose a rational expression with denominator of repeated linear factors into a sum of partial fractions.
- Decompose a rational expression with denominator of non-repeated irreducible quadratic factors into a sum of partial fractions.
- Decompose a rational expression with denominator of repeated irreducible quadratic factors into a sum of partial fractions.

Partial Fraction Decomposition

proper rational expression: degree of numerator < degree of denominator

Distinct Linear Factors

There are times, in future math classes, when you would like to break a rational expression into a sum of simpler fractions. We will begin with a proper fraction, where the degree of the numerator is less than the degree of the denominator. The first step is to factor the denominator and write it as a sum of n terms for an n^{th} degree denominator. $\frac{p(x)}{q(x)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3} + \cdots$

Ex 1: Determine A and B for this proper fraction. $\frac{3x-1}{x(x-4)} = \frac{A}{x} + \frac{B}{(x-4)}$ A, B constants

2 factors in denominator $\Rightarrow$ decompose into 2 fractions

$$\frac{3x-1}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$$

$$x(x-4)\frac{(3x-1)}{x(x-4)} = \frac{A}{\cancel{x}}\cancel{x}(x-4) + \frac{B}{\cancel{(x-4)}}x\cancel{(x-4)}$$

$$3x-1 = A(x-4) + Bx$$

note: this is true for all x, which means we can choose x-values we like.

goal: solve for A and B.

① $x=0$: $-1 = A(-4)$, $A = \frac{1}{4}$ ② $x=4$: $12-1 = 4B$, $B = \frac{11}{4}$

$$\Rightarrow \boxed{\frac{3x-1}{x(x-4)} = \frac{1/4}{x} + \frac{11/4}{x-4}}$$

If the fraction is improper, we $\boxed{\text{must}}$ do long division first.

Ex 2: Write the partial fraction decomposition for this expression. $\frac{x^2+1}{x^2-x}$

note: the degree of x^2+1 = degree of x^2-x

$\Rightarrow$ do long division first.

$$\begin{array}{r} 1 \\ x^2-x \enclose{longdiv}{x^2+1} \\ -(x^2-x) \\ \hline 1+x \end{array}$$

$$\frac{x^2+1}{x^2-x} = 1 + \underbrace{\frac{x+1}{x^2-x}}_{\text{proper rational expression}}$$

do PFD on $\frac{x+1}{x^2-x} = \frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$$\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$\cancel{x(x-1)}\frac{(x+1)}{\cancel{x(x-1)}} = \frac{A}{\cancel{x}}\cancel{x}(x-1) + \frac{B}{\cancel{x-1}}x\cancel{(x-1)}$$

$$x+1 = A(x-1) + Bx$$ true for all x; solve for A and B

① $x=0$: $1 = -A+0$, $A=-1$ ② $x=1$: $2 = 0+B$, $B=2$

$$\frac{x^2+1}{x^2-x} = 1 + \frac{x+1}{x(x-1)} = \boxed{1 + \frac{-1}{x} + \frac{2}{x-1}}$$

Repeated Linear Factors

$$\frac{p(x)}{q(x)} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \cdots + \frac{N}{(ax+b)^n}$$

Ex 3: Resolve into partial fractions $\dfrac{2x^2+7x+4}{(x+1)^3}$.

check: is our expression a proper rational expression? Yes.

$$(x+1)^3 \frac{2x^2+7x+4}{(x+1)^3} = \left[\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}\right](x+1)^3$$

$$2x^2+7x+4 = A(x+1)^2 + B(x+1) + C$$

note: true for all x; solve for A, B & C (not x). (choose any x-values)

① $x=-1$: $2(1)+7(-1)+4 = 0+0+C$

$-1 = C$

② $x=0$: $4 = A+B+(-1) \Leftrightarrow A+B=5$

③ $x=1$: $2+7+4 = A(4)+B(2)+(-1) \Leftrightarrow 14 = 4A+2B$

$7 = 2A+B$

② $B = 5-A$ ③ $7 = 2A+5-A$

$7 = 5+A$

$A=2 \Rightarrow B = 5-2 \Leftrightarrow B=3$

$$\boxed{\frac{2x^2+7x+4}{(x+1)^3} = \frac{2}{x+1} + \frac{3}{(x+1)^2} + \frac{-1}{(x+1)^3}}$$

Unique Irreducible Quadratic Factors

$$\frac{p(x)}{q(x)} = \frac{Ax+B}{a_1x^2+b_1x+c_1} + \frac{Cx+D}{a_2x^2+b_2x+c_2} + \cdots$$

irreducible: cannot be factored (w/ $\mathbb{R}$ factors)

Ex 4: Write the partial fraction decomposition of $\frac{-x^3+4x^2-2x+6}{x^2(x^2+2)}$.

check: is this a proper rational expression? Yes

deg(num) = 3, deg(den) = 4 $3<4$

⇒ do NOT need to do long division.

$$\frac{-x^3+4x^2-2x+6}{x^2(x^2+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2}$$

note: think of x as $(x-0)$. ⇒ x^2 is repeated linear factor.

$$\cancel{x^2(x^2+2)}\frac{(-x^3+4x^2-2x+6)}{\cancel{x^2(x^2+2)}} = \frac{A}{\cancel{x}}x^{\cancel{2}}(x^2+2) + \frac{B}{\cancel{x^2}}\cancel{x^2}(x^2+2) + \frac{(Cx+D)}{\cancel{x^2+2}}x^2\cancel{(x^2+2)}$$

$$-x^3+4x^2-2x+6 = Ax(x^2+2) + B(x^2+2) + (Cx+D)x^2$$

① $x=0$: $6 = 0+2B+0$

$B=3$

② $x=1$: $-1+4-2+6 = A(3)+3(3)+(C+D)$

$7 = 3A+9+C+D \iff -2 = 3A+C+D$

③ $x=-1$: $1+4+2+6 = -A(3)+3(3)+(-C+D)$

$13 = -3A+9-C+D \iff 4 = -3A-C+D$

add ② + ③

$$\begin{aligned} -2 &= 3A+C+D \\ +\quad 4 &= -3A-C+D \\ \hline 2 &= 2D \iff D=1 \end{aligned}$$

② $-3 = 3A+C$

③ $3 = -3A-C$ use substitution now: $C = -3-3A$

④ $x=2$: $-2^3+4(2^2)-2(2)+6 = A(2)(2^2+2)+3(2^2+2) + (C(2)+1)(2^2)$

$-8+16-4+6 = 12A+18+8C+4$

$10 = 12A+8C+22$

$-12 = 12A+8C \iff -3 = 3A+2C$

from ②

$-3 = 3A+2(-3-3A)$

$-3 = 3A-6-6A$

$3 = -3A$

$A=-1$

$\Rightarrow C = -3-3(-1)$

$C=0$

$\boxed{B=3, \; D=1}$

$$\Rightarrow \boxed{\frac{-x^3+4x^2-2x+6}{x^2(x^2+2)} = \frac{-1}{x} + \frac{3}{x^2} + \frac{1}{x^2+2}}$$

Repeated Irreducible Quadratic Factors

$$\frac{p(x)}{q(x)}=\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Ex 4: Write the partial fraction decomposition of $\frac{x^2+x+2}{(x^2+2)^2}$.

check deg(num)=2, deg(den)=4, 2<4 ✓

⇒ do NOT need to do long division.

$$(x^2+2)^2\,\frac{x^2+x+2}{(x^2+2)^2}=\left[\frac{Ax+B}{x^2+2}+\frac{Cx+D}{(x^2+2)^2}\right](x^2+2)^2$$

$$x^2+x+2=(Ax+B)(x^2+2)+(Cx+D)$$

① $x=0$: $2=B(2)+D \iff 2=2B+D$

② $x=1$: $1+1+2=(A+B)(3)+(C+D)$

$4=3A+3B+C+D$

③ $x=-1$: $1-1+2=(-A+B)(3)+(-C+D)$

$2=-3A+3B-C+D$

④ $x=2$: $4+2+2=(2A+B)(4+2)+(2C+D)$

$8=12A+6B+2C+D$

So we have 4 eqns and 4 unknowns

① $2B+D=2$

② $3A+3B+C+D=4$

③ $-3A+3B-C+D=2$

④ $12A+6B+2C+D=8$

will use Cramer's Rule to solve.

$$\begin{bmatrix}0&2&0&1\\3&3&1&1\\-3&3&-1&1\\12&6&2&1\end{bmatrix}=P$$

(coefficient matrix)

$\det(P)=12$

$$A=\frac{1}{12}\begin{vmatrix}2&2&0&1\\4&3&1&1\\2&3&-1&1\\8&6&2&1\end{vmatrix}=\frac{1}{12}(0)=0 \qquad B=\frac{1}{12}\begin{vmatrix}0&2&0&1\\3&4&1&1\\-3&2&-1&1\\12&8&2&1\end{vmatrix}=\frac{1}{12}(12)=1$$

$$C=\frac{1}{12}\begin{vmatrix}0&2&2&1\\3&3&4&1\\-3&3&2&1\\12&6&8&1\end{vmatrix}=\frac{1}{12}(12)=1 \qquad D=\frac{1}{12}\begin{vmatrix}0&2&0&2\\3&3&1&4\\-3&3&-1&2\\12&6&2&8\end{vmatrix}=\frac{1}{12}(0)=0$$

$$\Rightarrow \boxed{\frac{x^2+x+2}{(x^2+2)^2}=\frac{1}{x^2+2}+\frac{x}{(x^2+2)^2}}$$