Math 1050 ~ College Algebra

26 Systems of Linear Equations: Determinants

Determinant of a Matrix

Every square matrix has a number associated with it, called the determinant of A. It may be written $\det(A)$ or $|A|$.

For a 2×2 matrix, $\det(A)$ is given by this formula.

$$\text{Det}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

EX 1

Find the determinant of each of these matrices.

1a)

$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

1b)

$\begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix}$

1c)

$\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

Cramer's Rule

For a set of two equations in two unknowns, Cramer's Rule says that

$$\begin{matrix} ax + by = c \\ dx + ey = f \end{matrix} \text{ has solutions } x = \frac{ce - bf}{ae - bd}, y = \frac{af - cd}{ae - bd}.$$

EX 2

Use the rule above to determine the solution.

$$2x + y = 4$$
$$5x + 3y = -1$$

Determinant of a 3×3 matrix is more complex.

$$A= \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \det(A) = |A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\text{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Given the square $n \times n$ matrix A where $n > 1$, and a_{ij} represents the entry in the i^{th} row and j^{th} column:

- the <u>minor</u>, M_{ij} of the entry a_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix left after deleting row i and column j from the matrix A.
- the cofactor, C_{ij} of entry a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.

EX 3

Find all M_{ij} and C_{ij} for this matrix. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

The determinant of an $n \times n$ matrix, where $\text{n} > 1$, is the sum of the entries in any row or column multiplied by each entry's respective cofactor.

EX 4

Find the determinant of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$.

To use Cramer's Rule to solve a set of 3 equations, let $D = \det A$. D_x is found by replacing the first column of A by the constants. D_y is found by replacing the second column of A by the constants, and D_z is found by replacing the third column of A by the constants.

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$$

EX 5

Use Cramer's Rule to solve.

$$\begin{aligned} x - y &= 1 \\ x - z &= -2 \\ 6x - 2y - 3z &= -4 \end{aligned}$$