2022 State Math Competition

Junior Exam

Version A

Instructions:

- Make sure to mark the version on your answer sheet.
- Correct answers are worth 5 points. Unanswered questions will be given 2 points. Incorrect answers will be worth 0 point. This means that it is not in your best interest to guess answers unless you have eliminated some possibilities.
- Fill in the answers on the answer sheet using a pencil or pen.
- Time limit: 75 minutes.
- When you are finished, please give the exam and any scrap paper to the test administrator.
- Good luck!

1. A company manufactures boxes that are 10 cm long, 8 cm wide, and 2 cm high. If the height of the box is increased by 50%, and the width is decreased by 25%, what is the percent increase in the volume of the box?

   (A) 5%
   (B) 10%
   (C) 20%
   (D) 37.5%
   (E) 12.5%

   Correct answers: (E)

   Explanation: Originally the box has volume $10 \cdot 8 \cdot 2 = 160$ cm^3. The height is increased 50% to 3 cm, and the width is decreased by 25% to 6 cm. The new volume is $10 \cdot 6 \cdot 3 = 180$ cm^3. This is an increase of $\frac{20}{160} = 12.5\%$.

2. It is well known that Bojan is dangerous from within the paint as well as from behind the three-point arc. The number of 2-point shots he made last night was $\frac{5}{3}$ times as many as the number of 3-point shots he made. He also made 4 free throws (each worth 1 point). Knowing that he scored a total of 23 points, how many 3-point shots did he make?

   (A) 4
   (B) 1
   (C) 5
   (D) 2
   (E) 3

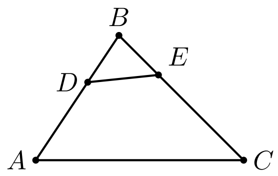
   Correct answers: (E)

   Explanation: Let t be the number of three-pointers Bojan made. Then

   $$\frac{5}{3} \cdot 2t + 3t + 4 = 23.$$

   This simplifies to $\frac{19}{3}t = 19$, so $t = 3$.

3. In $\triangle ABC$, $\angle A = 35°$ and $\angle C = 65°$. D is on the side $\overline{AB}$, and E is on the side $\overline{BC}$ so that $BD = BE$. What is the measure of $\angle BDE$?



(A) $30°$
(B) $50°$
(C) $35°$
(D) $45°$
(E) $40°$

Correct answers: (B)

Explanation: $\angle A + \angle B + \angle C = 180°$, so $\angle B = 80°$. $\triangle BDE$ is isosceles since $BD = BE$, so the opposite angles $\angle BDE = \angle BED$. Then $\angle BDE + \angle BED + \angle B = 180°$ becomes $2\angle BDE + 80° = 180°$, so $\angle BDE = 50°$.

4. For a natural number n, we define the factorial of n by $n! = n(n-1)(n-2)\cdots 1$. For example $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. The number $1! + 2! + 3! + \ldots + 2022!$ has one's digit equal to

(A) 3
(B) 2
(C) 8
(D) 4
(E) 9

Correct answers: (A)

Explanation: If $n \geq 5$, then $n!$ contains a factor of 2 and 5, so the one's digit is 0. So we only need to add up the first four terms

$$1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33.$$

The one's digit is 3.

5. The ratio $\dfrac{10^{2020} + 10^{2022}}{10^{2021} + 10^{2021}}$ is closest to which of the following numbers?

(A) 5
(B) 1
(C) 10
(D) 9
(E) 0.5

Correct answers: (A)

Explanation: We factor

$$\frac{10^{2020} + 10^{2022}}{10^{2021} + 10^{2021}} = \frac{10^{2020}(1 + 10^2)}{10^{2020}(10 + 10)} = \frac{101}{20}.$$

As $\frac{101}{20} = 5.05$, our answer is 5.

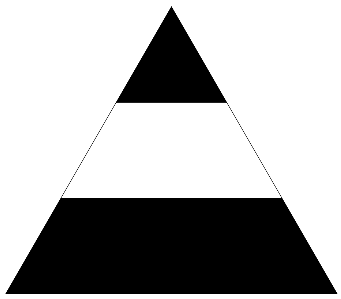
6. Suppose we have two glasses of equal volume, glass A and glass B. Glass A is filled with milk while glass B is filled with water. One spoon of milk is transferred to glass B. Glass B is stirred, then one spoon of the mixture in glass B is transferred to glass A. Which of the following is true?

   (A) The volume of milk in glass B is smaller than the volume of water in glass A.
   (B) The information is insufficient to determine the answer.
   (C) The volume of milk in glass B is equal to the volume of water in glass A.
   (D) The volume of milk in glass B is bigger than the volume of water in glass A.
   (E) The volume of milk in glass A is smaller then the volume of water in glass A.

   Correct answers: (C)

   Explanation: The volumes of the glasses and spoon are irrelevant for this problem. For the sake of the explanation, suppose the volume of the glasses is 1 liter, while the volume of the spoon is x liters, with $0 < x < 1$. Once we take a spoonful of milk from glass A and put it into glass B and stir, glass A contains $1 - x$ liters of milk, while glass B contains $1 + x$ liters of liquid at a concentration of $\frac{x}{1+x}$ liters of milk per liter of water. A spoonful of this mixture will contain $x(\frac{x}{1+x}) = (\frac{x^2}{1+x})$ liters of milk. So after this spoonful is put back in glass A, glass B will have $x - \frac{x^2}{1+x}$ liters of milk in it. Then amount of water in spoonful is $x(1 - \frac{x}{1+x})$ liters, and that is the amount of wather glass A will have in it. So the volume of milk in glass B is equal to the volume of water in glass A.

7. Consider an equilateral triangle with height 3. Cut it into three strips of equal height as shown in the figure below and color the top and bottom strip black. What is the total area of the black regions?



   (A) $2\sqrt{3}$
   (B) $\frac{\sqrt{3}}{2}$
   (C) $3\sqrt{2}$
   (D) $\frac{3}{\sqrt{2}}$
   (E) None of the above

   Correct answers: (A)

   Explanation: The black triangle on top may be rotated and fit together with the black trapezoid on the bottom to create a parallelogram:

   The height of this parallelogram is 1 so it just remains to calculate its base. The base is the side length of the original equilateral triangle. Our equilateral triangle is similar to the equilateral triangle with side length 1 which has height $\frac{\sqrt{3}}{2}$. So the base b of our triangle satisfies

$$\frac{b}{1} = \frac{3}{\sqrt{3}/2} = 2\sqrt{3}.$$

   Finally then our black parallelogram has area $2\sqrt{3}$.

8. Given that $i^2 = -1$, find $(1+i)^{10} =$

(A) 0
(B) -32
(C) $32i$
(D) 32
(E) $-32i$

Correct answers: (C)

Explanation: Rewrite in polar coordinate, $1+i = \sqrt{2}e^{i\pi/4}$. We raise it to the 10th power, and get $(\sqrt{2}e^{i\pi/4})^{10} = 32e^{i5\pi/2} = 32i$. So the answer is $32i$.

Another way to do this that does not involve DeMoivre's theorem is to realize that

$$(1+i)^2 = 1+2i-1 = 2i$$

So

$$(1+i)^{10} = (2i)^5 = 2^5 i^5 = 32i$$

9. How many four-digit numbers of the form $abab$ are divisible by 16?

(A) 1
(B) 4
(C) 3
(D) 2
(E) more than 4

Correct answers: (E)

Explanation: Numbers of the form $\overline{abab}$ can be written as

$$1000a + 100b + 10a + b = 101(10a+b).$$

Since 101 is prime, this is divisible by 16 if and only if $10a+b$ is divisible by 16. So it turns out that $abab$ is divisible by 16 if and only if ab is. There are exactly six possibilities for (a,b): $(1,6)$, $(3,2)$, $(4,8)$, $(6,4)$, $(8,0)$, $(9,6)$.

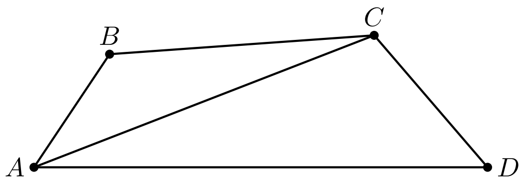
10. Let a, b, c be real numbers such that $c(a+b+c) < 0$. Which of these is a possible value of $b^2 - 4ac$?

(A) -37
(B) 94
(C) 0
(D) -12
(E) all of the above

Correct answers: (B)

Explanation: Consider the quadratic function $f(x) = ax^2 + bx + c$. The given condition is equivalent to $f(0) \cdot f(1) < 0$. It follows that $f(0)$ and $f(1)$ have opposite signs. This implies that $f(x)$ has a zero in the interval $(0,1)$. Since f has a real zero (and attains both positive and negative values), the discriminant must be positive, so $b^2 - 4ac > 0$. One verifies easily that any positive discriminant is possible.

11. In quadrilateral $ABCD$, $AB = 6$, $BC = 10$, $CD = 7$, and $DA = 21$. If AC is an integer, then what is AC?



(A) 14
(B) 15
(C) 17
(D) 16
(E) 18

Correct answers: (B)

Explanation: Applying the triangle inequality to $\triangle ABC$ gives $AC < 6 + 10 = 16$. Applying the triangle inequality to triangle ACD gives $AC + 7 > 21$ so $AC > 14$. Since AC is an integer, it must be that $AC = 15$.

12. A sequence is generated using this procedure. The first number in the sequence is 1. To generate each successive term, a fair coin is flipped. If the result is heads, the next term is found by first multiplying the previous term by 2, and then subtracting 1. If the result is tails, the next term is found by first multiplying the previous term by 3, and then subtracting 1. What is the probability that the fourth term in the sequence is even?

(A) $\frac{3}{8}$
(B) $\frac{2}{3}$
(C) $\frac{1}{4}$
(D) $\frac{1}{6}$
(E) $\frac{1}{2}$

Correct answers: (A)

Explanation: We need 3 flips to get the second, third, and fourth terms in the sequence. We can make a tree of the possibilities.

$$1 \begin{cases} \text{H} : 1 & \begin{cases} \text{H} : 1 & \begin{cases} \text{H} : 1 \\ \text{T} : 2 \end{cases} \\ \text{T} : 2 & \begin{cases} \text{H} : 3 \\ \text{T} : 5 \end{cases} \end{cases} \\ \text{T} : 2 & \begin{cases} \text{H} : 3 & \begin{cases} \text{H} : 5 \\ \text{T} : 8 \end{cases} \\ \text{T} : 5 & \begin{cases} \text{H} : 9 \\ \text{T} : 14 \end{cases} \end{cases} \end{cases}$$

Each of the 8 outcomes for the fourth number in the sequence is equally likely. Out of those 8, 3 are even (corresponding to HHT, THT, and TTT). The probability is $\frac{3}{8}$.

13. In how many ways, we an rearrange the letters $BANANAS$?

(A) 420
(B) 240
(C) 360
(D) 600
(E) 1260

Correct answers: (A)

Explanation: A standard counting gives

$$\frac{7!}{3!2!1!1!} = 420.$$

In the denominator, $3!$ corresponds to the three A's being identical, and $2!$ corresponds to the two N's being identical.

14. The sides a, b, c of a triangle satisfy the relations $c^2 = 2ab$ and $a^2 + c^2 = 3b^2$. Then, $\angle BAC =$

(A) 45
(B) 90
(C) 60
(D) 30
(E) impossible to determine

Correct answers: (A)

Explanation: We have $a^2 + 2ab - 3b^2 = 0 \implies (a-b)(a+3b) = 0$.
So,$a = b, c = a\sqrt{2} \implies \angle B = 45$.

15. A new sequence is obtained from the sequence of the positive integers by deleting all the perfect squares. The 2022nd term of the new sequence is

(A) 2067
(B) 2064
(C) 2066
(D) 2068
(E) 2065

Correct answers: (A)

Explanation: Since $[\sqrt{2022}] = 44$ and $45^2 = 2025, 46^2 = 2116$, so the required number is $2022 + 45 = 2067$.

16. What is the coefficient of x^3 in the product $(x+1)(x+2)(x+2)(x+3)(x+3)$?

(A) 47
(B) 39
(C) 97
(D) 38
(E) 27

Correct answers: (A)

Explanation: We can obtain a x^3 term from the product by choosing an x out of three of the parentheses and a number out of the remaining two parentheses. There are $\binom{5}{3} = 10$ ways to do this giving x^3 terms with coefficients 2 (the product of the numbers from the first two parentheses), 2 (the product of the numbers from the first and third parentheses), 3 (the product of the numbers from the first and fourth parentheses), etc. Adding up all the possibilities gives the coefficient

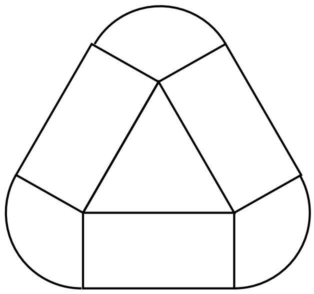
$$2 + 2 + 3 + 3 + 4 + 6 + 6 + 6 + 6 + 9 = 47.$$

17. An equilateral triangle has side length 2. What is the area of the region that contains all points outside the triangle but not more than 1 unit away from a point in the triangle?

(A) $6+\sqrt{3}\pi$
(B) $3+3\sqrt{2}$
(C) $6+\pi$
(D) $6+\sqrt{3}$
(E) $3+3\pi$

Correct answers: (C)

Explanation: There are two cases for how to find the distance from a point outside the triangle to the triangle. If the point is above one of the sides, then we can drop a perpendicular segment to the side of the triangle. If the point is near a corner of the triangle, the perpendicular lines miss the sides of the triangle, and we calculate the distance from the corner of the triangle instead. The first case corresponds to rectangular regions above each side, and the second case corresponds to sectors (portions of circles) on each corner. The 3 rectangles each have area $2 \cdot 1 = 2$. Since the triangle has angles $60°$ and the rectangles have right angles, the angle of each sector is $360° - 60° - 90° - 90° = 120°$. In other words, each sector is a third of a circle of radius 1. Putting the 3 sectors together, we have an area of $\pi \cdot 1^2 = \pi$. The total area is $6 + \pi$.



18. Seven points are selected on a circle. Three of the chords joining pairs of points are selected at random. What is the probability that the three chords form a triangle?

(A) $\frac{1}{38}$
(B) $\frac{1}{21}$
(C) $\frac{3}{49}$
(D) $\frac{2}{9}$
(E) $\frac{1}{19}$

Correct answers: (A)

Explanation: There are $\binom{7}{2} = \frac{7\cdot 6}{2} = 21$ total chords joining pairs of points. So there are $\binom{21}{3}$ total ways to choose 3 of the chords. Now we need to determine how many of these combinations form a triangle. For each set of 3 of the points, there is exactly one triangle with those vertices. So there are $\binom{7}{3}$ total ways for the chords to form a triangle. The probability is

$$\frac{\binom{7}{3}}{\binom{21}{3}} = \frac{\frac{7\cdot 6\cdot 5}{3!}}{\frac{21\cdot 20\cdot 19}{3!}} = \frac{7\cdot 6\cdot 5}{21\cdot 20\cdot 19} = \frac{1}{38}.$$

19. $1^2 - 2^2 + 3^2 - 4^2 + ... + (101)^2 - (102)^2$ is equal to

(A) -4851
(B) -5253
(C) -5050
(D) -4685
(E) -5464

Correct answers: (B)

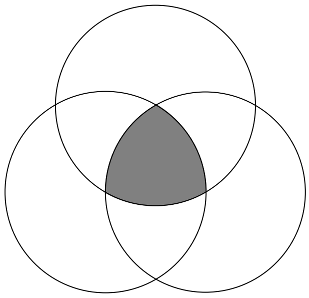
Explanation:

$$\begin{aligned}
&1^2 - 2^2 + 3^2 - 4^2 + \cdots + (101)^2 - (102)^2 \\
=&(1^2 - 2^2) + (3^2 - 4^2) + \cdots + (101)^2 - (102)^2) \\
=&(1-2)(1+2) + (3-4)(3+4) + \cdots + (101-102)(101+102)) \\
=&-1[3 + 7 + 11 + \cdots + 203].
\end{aligned}$$

Now $3 + 7 + 11 + \cdots + 203$ is an arithmetic series with common difference 4. Note that

$$3 + 7 + 11 + \cdots + 203 = \sum_{n=1}^{51}(-1+4n) = -51 + 4\frac{(51)(52)}{2} = 5253$$

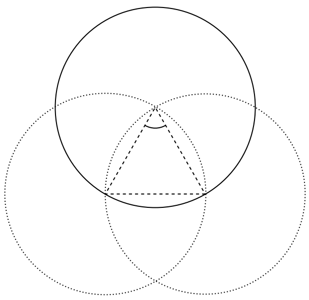
20. Consider three circles of radius 1 which pass through each other's centers as shown in the figure below. What is the area of the shaded region enclosed by all of them?



(A) $\frac{\pi}{4} - \frac{\sqrt{3}}{4}$
(B) $\frac{\sqrt{3}}{4}$
(C) $\frac{\pi}{2} - \frac{\sqrt{3}}{2}$
(D) $\frac{\pi}{2} - \frac{\sqrt{3}}{4}$
(E) None of the above

Correct answers: (C)

Explanation: It will be helpful to focus on one of the circles (say the one in the middle in the figure) and to refer to the figure below:



Draw in a triangle in the middle of the shaded region as shown. This is an equilateral triangle. The shaded region consists of this equilateral triangle plus 3 "slivers." The side lengths of the equilateral triangle are all 1. By the Pythagorean theorem it has base 1 and height $\frac{\sqrt{3}}{2}$. Hence its area is $\frac{\sqrt{3}}{4}$. The angle γ shown is $\pi/3$. Hence the sector of the middle circle cut off by the angle γ is $\frac{1}{2}\frac{\pi}{3}(1)^2 = \frac{\pi}{6}$. This sector is the equilateral triangle plus one sliver so a single sliver has area $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$. Finally then, the area of the shaded region is the area of the triangle plus the areas of the three slivers:

$$\frac{\sqrt{3}}{4} + 3\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{2} - \frac{\sqrt{3}}{2}.$$