State Math Contest 2019 Senior Exam
Weber State University
March 5, 2019

Instructions:

- Do not turn this page until your proctor tells you.
- Enter your name, grade, and school information following the instructions given by your proctor.
- Calculators are not allowed on this exam.
- This is a multiple choice test with 40 questions. Each question is followed by answers marked (a), (b), (c), (d), and (e). Only one answer is correct.
- Mark your answer to each problem on the bubble sheet Answer Form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.
- Scoring: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- You will have 2 hours and 30 minutes to finish the test.
- You may not leave the room until at least 10:15 a.m.

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1. Using each of the nine digits (not 0) only once, find the smallest sum that primes, created from the nine digits, can form. For example, 61 + 283 + 47 + 59 = 450.

(a) 113
(b) 207
(c) 321
(d) 431
(e) None of the above

Correct Answer: (c) 4,096.
Solution: There are exactly 12 prime numbers under 40 (i.e., 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37), and we have $m = 2 \times 3 \times 5 \times \cdots \times 31 \times 37$. Each divisor of m is a product of the form $2^a \times 3^b \times 5^c \times \cdots \times 31^k \times 37^l$, where $a, b, c, \ldots, k, l$ are either 0 or 1. There are two choices (0 or 1) for each of the 12 powers, so the total number of positive divisors is $2^{12} = 4{,}096$.

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2. Let $f(x) = ax^2 + bx + c$ be a quadratic function with two real roots, x_1 and x_2. If $f(x_1) = p$, then $f'(x_2)$ is:

(a) p
(b) $-p$
(c) $2ap + b$
(d) $-2ap + b$
(e) None of the above

Correct Answer: (b) $-p$
Solution: By the quadratic formula, $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ and $f'(x) = 2ax + b$. Hence, if
$f'(x_1) = 2ax_1 + b = 2a \cdot \frac{-b+\sqrt{b^2-4ac}}{2a} + b = \sqrt{b^2 - 4ac} = p$, then
$f'(x_2) = 2ax_2 + b = 2a \cdot \frac{-b-\sqrt{b^2-4ac}}{2a} + b = -\sqrt{b^2 - 4ac} = -p$.

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3. What is the coefficient of the term which contains y^4 in the expansion of $(2x^2 - \sqrt{y})^{14}$?

(a) 192,192
(b) 256
(c) –439,296
(d) –262,144
(e) 1,025,024

Correct Answer: (a) 192,192
Solution: To get a term in the expansion with y^4 we must have the second term of the given expansion to be raised to the 8th power $(\sqrt{y})^8 = y^4$. If the $\sqrt{y}$ is raised to the 8th power, then that means $2x^2$ is raised to the 6th power (the sum of the powers applied to each term for the expansion must be 14). If the $\sqrt{y}$ is raised to the 8th power, then the coefficient of this term is $\binom{14}{8}$. Now simplify the term to find the coefficient:

$\binom{14}{8}(2x^2)^6(-\sqrt{y})^8$
$= \frac{14!}{8!(14-8)!}(2x^2)^6(-\sqrt{y})^8$
$= \frac{14\cdot13\cdot12\cdot11\cdot10\cdot9\cdot8!}{8!\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}(2x^2)^6(-\sqrt{y})^8$
$= \frac{14\cdot13\cdot12\cdot11\cdot10\cdot9}{6\cdot5\cdot4\cdot3\cdot2\cdot1}(2x^2)^6(-\sqrt{y})^8$
$= \frac{14\cdot13\cdot11\cdot9}{6\cdot1}(2x^2)^6(-\sqrt{y})^8$
$= 3003\cdot(2x^2)^6(-\sqrt{y})^8$
$= 3003\cdot2^6\cdot x^6(-\sqrt{y})^8$
$= 3003\cdot2^6\cdot x^6y^4$
$= 3003\cdot64\cdot x^6y^4$
$= 192,192x^6y^4$

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4. A jar with one marble sits on a desk. The first student passes the jar and adds a marble. The second student passes the jar and adds two marbles. The third student adds three marbles. Each student that passes adds one more marble to the jar than the previous student added. How many marbles will be in the jar after 100 students have added marbles to the jar?

(a) 101
(b) 501
(c) 1,001
(d) 5,051
(e) 10,101

Correct Answer: (d) 5,051
Solution: The first student puts in 1 marble, the second puts in 2, until the hundredth student puts in 100 marbles. The following equation models the situation:

Total = 1 + 1 + 2 + 3 + 4 + … + 99 + 100

One way to find the sum of 1 + 2 + … + 100 is as follows. Let's add two sets of marbles together in a strategic way and then divide the total by two:

$T = 1 + 2 + 3 + 4 + \ldots + 99 + 100$
$T = 100 + 99 + 97 + 96 + \ldots + 2 + 1$

Adding each *column* of the equation gives:

$2T = 101 + 101 + 101 + 101 + \ldots + 101 + 101$ (100 times)
$2T = 100 \times 101$
$T = 50 \times 101$
$T = 5{,}050$

Then we account for the original marble in the jar, as stated in the first sentence, and we have 5,051 marbles in the jar.

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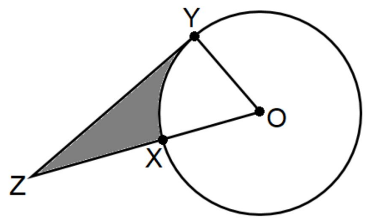
5. In the figure below, O is the center of the circle, $\overline{YZ}$ is tangent to the circle at Y, and X lies on $\overline{OZ}$. If $\overline{OX} = \overline{XZ} = 6$, what is the area of the shaded region?



(a) $18\sqrt{3} - 3\pi$
(b) $18\sqrt{3} - 6\pi$
(c) $36\sqrt{3} - 3\pi$
(d) $36\sqrt{3} - 6\pi$
(e) $18\sqrt{3} - 2\pi$

Correct Answer: (b) $18\sqrt{3} - 6\pi$
Solution: Since $\overline{YZ}$ is tangent to the circle at Y, it follows that $\overline{YZ} \perp \overline{OY}$, and so $\triangle OYZ$ is a right triangle with its right angle at Y. Since $\overline{OY} = 6$ and $\overline{OY}$ and $\overline{OX}$ are both radii of the circle, $\overline{OX} = \overline{OY} = 6$, and $\overline{OZ} = \overline{OX} + \overline{XZ} = 12$. Thus, $\triangle OYZ$ is a right triangle with the length of the hypotenuse ($\overline{OZ} = 12$) twice the length of one of its legs ($\overline{OY} = 6$). It follows that $\triangle OYZ$ is a 30°-60°-90° triangle with its 30° angle at Z and its 60° angle at O. The area of the shaded region is the area of $\triangle OYZ$ minus the area of the sector bounded by radii $\overline{OX}$ and $\overline{OY}$.

In the 30°-60°-90° $\triangle OYZ$, the length of $\overline{OY}$, which is opposite the 30° angle, is 6. Thus, the length of $\overline{YZ}$, which is opposite the 60° angle, is $6\sqrt{3}$. Hence, the area of $\triangle OBC$ is $\frac{1}{2}(6)\left(6\sqrt{3}\right) = 18\sqrt{3}$. Since the sector bounded by radii $\overline{OX}$ and $\overline{OY}$ has central angle 60°, the area of this sector is $\frac{60}{360} = \frac{1}{6}$ of the area of the circle. Since the circle has radius 6, its area is $\pi(6)^2 = 36\pi$, and so the area of the sector is $\frac{1}{6}(36\pi) = 6\pi$. Therefore, the area of the shaded region is $18\sqrt{3} - 6\pi$.

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6. How many different numbers can you get by adding three different numbers from the set {3, 6, 9, 12, …, 21, 24}?

(a) 15
(b) 16
(c) 18
(d) 20
(e) 22

Correct Answer: (b) 16
Solution: The smallest such sum is $3 + 6 + 9 = 18$ and the largest is $18 + 21 + 24 = 63$. All sums are multiples of three and you can get all from 18 to 63 with different numbers. The number of multiples between 18 and 63 inclusive is 16.

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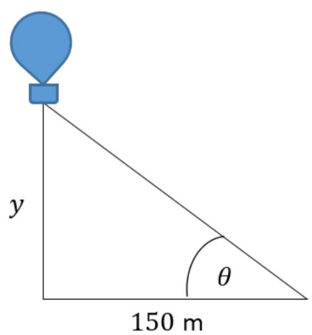
7. A hot air balloon rising straight up from a level field is tracked by a rangefinder 150 meters from the liftoff point. At the moment the rangefinder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.15 radians/minute. How fast is the balloon rising at that moment?

(a) $22.5\sqrt{2}$ meters/min
(b) 45 meters/min
(c) $\frac{45}{\sqrt{2}}$ meters/min
(d) 150 meters/min
(e) 0.3 meters/min

Correct Answer: (b) 45 meters/min
Solution:



We are asked to find $\frac{dy}{dt}$ when $\theta = \frac{\pi}{4}$ and $\frac{d\theta}{dt} = 0.15$ radians/min. First, relate y and θ by the equation $\tan\theta = \frac{y}{150}$. Differentiating both sides with respect to t yields $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{150} \cdot \frac{dy}{dt}$. Substituting the values for θ and $\frac{d\theta}{dt}$ gives $\frac{dy}{dt} = 45$ meters/min.

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8. Given that $x^2 + y^2 = 9$, what is the largest possible value of $x^2 + 3y^2 + 4x$?

(a) 22
(b) 24
(c) 26
(d) 29
(e) 36

Correct Answer: (d) 29
Solution: Write $y^2 = 9 - x^2$, and substitute it into the other formula to get $x^2 + 3(9 - x^2) + 4x = -2x^2 + 4x + 27 = -2(x - 1)^2 + 29$, where we can see the max is 29.

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9. A 50-inch piece of wire is cut into two pieces which are then bent into a square and a circle. How long should the two pieces be in order to minimize the sum of the areas?

(a) 25 inches, 25 inches
(b) 20 inches, 30 inches
(c) 38 inches, 12 inches
(d) 4 inches, 46 inches
(e) 22 inches, 28 inches

Correct Answer: (e) 22 inches, 28 inches
Solution: Call the length where you will cut the wire x. Then the other piece is $50 - x$. You can let either piece be the perimeter of the square. Choose $50 - x$. Then each side of the square is $\frac{50-x}{4}$, which means the area of the square is $\left(\frac{50-x}{4}\right)^2$. The circumference of the circle is then equal to x and we can solve that equation for r to get $r = \frac{x}{2\pi}$. Thus, the sum of the areas is $A = \left(\frac{50-x}{4}\right)^2 + \pi\left(\frac{x}{2\pi}\right)^2 = \left(\frac{50-x}{4}\right)^2 + \frac{x^2}{4\pi}$. Differentiation with respect to x yields $A' = \frac{x}{2\pi} - \frac{50-x}{8}$. Setting the derivative equal to zero and solving for x gives $x \approx 22$ inches. Thus, the other piece must be $50 - 22 = 28$ inches long.

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10. A **cycloid** is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping.

One arch of a cycloid generated by a circle of radius r can be parameterized by

$x = r(t - \sin t)$
$y = r(1 - \cos t)$,

where t is the angle through which the rolling wheel has rotated.

Find the area under one cycloid cycle for a wheel with radius 18 inches.

(a) 4π feet squared
(b) $\frac{21\pi}{4}$ feet squared
(c) 6π feet squared
(d) $\frac{27\pi}{4}$ feet squared
(e) 8π feet squared

Correct Answer: (c) $\frac{27\pi}{4}$ feet squared
Solution:

$$A = \int_0^{2\pi r} y\, dx$$

$$= \int_0^{2\pi r} r(1 - \cos t)\, dx$$

$$= \int_0^{2\pi} r(1 - \cos t) \frac{dx}{dt} dt$$

$$= \int_0^{2\pi} r(1 - \cos t) r(1 - \cos t) dt$$

$$= r^2 \int_0^{2\pi} (1 - \cos t)^2 dt$$

$$= r^2 \int_0^{2\pi} (1 - 2\cos t + cos^2 t)\, dt$$

$$= r^2 \int_0^{2\pi} (1 - 2\cos t + \frac{1}{2} + \frac{\cos 2t}{2})\, dt$$

$$= r^2 [t + 2\sin t + \frac{t}{2} + \frac{\sin 2t}{4}]_0^{2\pi}$$

$$= 3\pi r^2$$

$$= 3\pi (\frac{3}{2})^2 \text{ sq ft}$$

$$= \frac{27\pi}{4} \text{ sq ft}$$

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11. Let $log_b x = 2$, $log_b y = 4$, and $log_b z = 8$. For some positive values of a, b, x, y, and z, what is the exact value of $\left(log_{a^4} \frac{x^2 y^3}{\sqrt{z}}\right) log_b a$?

(a) 1
(b) $\frac{256}{\sqrt{8}}$
(c) –7
(d) 3
(e) None of the above

Correct Answer: (d) 3
Solution: Observe that, due to the properties of logarithms,

$$log_{a^4} \frac{x^2 y^3}{\sqrt{z}} \cdot log_b a = \frac{1}{4}\left(2 \cdot log_a x + 3 \cdot log_a y - \frac{1}{2} \cdot log_a z\right) \cdot log_b a$$
$$= \frac{1}{2} \cdot log_a x \cdot log_b a + \frac{3}{4} \cdot log_a y \cdot log_b a - \frac{1}{8} \cdot log_a z \cdot log_b a$$

$= \frac{1}{2} \cdot log_b a^{log_a x} + \frac{3}{4} \cdot log_b a^{log_a y} - \frac{1}{8} log_b a^{log_a z}$
$= \frac{1}{2} \cdot log_b x + \frac{3}{4} \cdot log_b y - \frac{1}{8} \cdot log_b z$
$= \frac{1}{2} \cdot 2 + \frac{3}{4} \cdot 4 - \frac{1}{8} \cdot 8$
$= 3$

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12. **Trailing zeros** are a sequence of zeros after which no other nonzero digits follow. Determine how many trailing zeros are in (130!)

(a) 13
(b) 26
(c) 31
(d) 32
(e) 35

Correct Answer: (d) 32
Solution: Each factor of five adds another zero to the product. There are 26 factors in 130! that are divisible by five. There are an additional five factors that are divisible by 52. There is yet another factor that is divisible by 53. Altogether there are 32 factors of five, and thus 32 trailing zeros in 130!

Just for fun, according to Wolfram Alpha, 130! equals

6 466 855 489 220 473 672 507 304 395 536 485 253 155 359 447 828 049 608 975 952 ⋱
322 944 781 961 185 526 165 512 707 047 229 268 452 925 683 969 240 398 027 149 ⋱
120 740 074 042 105 844 737 747 799 459 310 029 635 780 991 774 612 983 803 150 ⋱
965 145 600 000 000 000 000 000 000 000 000 000 000

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13. What are the dimensions of a right circular cylinder can that will contain 1 cubic foot of liquid and that will have the smallest amount of surface area?

(a) The ideal radius is between 0.3 feet and 0.4 feet
(b) The ideal radius is between 0.4 feet and 0.5 feet
(c) The ideal radius is between 0.5 feet and 0.6 feet
(d) The ideal radius is between 0.6 feet and 0.7 feet
(e) None of above

Correct Answer: (c) The ideal radius is between 0.5 feet and 0.6 feet
Solution: The volume of the can is $V = 1\ \text{ft}^3 = \pi r^2 h$, and we can set $h = \frac{1}{\pi r^2}$. The surface area is $SA = 2\pi r^2 + h \cdot \text{circumference} = 2\pi r^2 + h \cdot 2\pi r$. To minimize the surface area, we write the equation in terms of r: $SA = 2\pi r^2 + \frac{1}{\pi r^2} \cdot 2\pi r = 2\pi r^2 + \frac{2}{r}$. From a table of the equation, we can find the ideal radius when the graph of the surface area attains a minimum, or when the graph of the surface area goes from decreasing to increasing.
Table:

Radius (r)	Surface Area $\left(2\pi r^2 + \frac{2}{r}\right)$	
0.1	20.063	
0.2	10.251	decreasing
0.3	7.232	decreasing
0.4	6.005	decreasing
0.5	**5.570**	**decreasing**
0.6	**5.594**	**increasing**
0.7	5.934	increasing
0.8	6.519	increasing

Or, we could use calculus. If we set the derivative of the surface area equation equal to zero, then we have the minimum radius, and thus we can find the minimum surface area. Since $SA = 2\pi r^2 + \frac{2}{r}$, then $\frac{dSA}{dr} = 4\pi r - \frac{2}{r^2}$. If we set it equal to 0 and simplify, we have:

$0 = 4\pi r - \frac{2}{r^2}$

$\frac{2}{r^2} = 4\pi r$

$2 = 4\pi r^3$

$\frac{1}{2} = \pi r^3$

$\frac{1}{2\pi} = r^3$

$r = \sqrt[3]{\frac{1}{2\pi}}$

$r = \frac{1}{\sqrt[3]{2\pi}}$

Since our answers are in ranges, we can use a basic approximation for pi, say 3. Then we'll have $r = \frac{1}{\sqrt[3]{6}}$, which is just over $\frac{1}{2}$, or between 0.5 and 0.6.

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14. A certain Archimedean solid built from squares and triangles, known as a rhombicuboctahedron, has 24 vertices and 48 edges. How many faces does it have?

(a) 72
(b) 1152
(c) 2
(d) 26
(e) 36

Correct Answer: (d) 26
Solution: We know from Euler's formula that $V - E + F = 2$, so we have $24 - 48 + F = 2$. This forces $F = 26$.

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15. What is the exact value of $\cos(\sec^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(-\frac{1}{4}\right))$?

(a) $\frac{3}{2} - \frac{1}{4}$
(b) $\frac{8\sqrt{17}+\sqrt{5}\sqrt{17}}{51}$
(c) $\frac{\sqrt{5}\sqrt{17}-8\sqrt{17}}{51}$
(d) $\frac{8\sqrt{17}-\sqrt{5}\sqrt{17}}{51}$

(e) None of the above

Correct Answer: (b) $\frac{8\sqrt{17}+\sqrt{5}\sqrt{17}}{51}$

Solution: If $\sec^{-1}\left(\frac{3}{2}\right) = \alpha$ and $\tan^{-1}\left(-\frac{1}{4}\right) = \beta$, then $\sec\alpha = \frac{3}{2}$ and $\tan\beta = -\frac{1}{4}$. This implies that $\cos\alpha = \frac{2}{3}$ and $\sin\alpha = \sqrt{1-cos^2\alpha} = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$, since the angle α is in the first quadrant. And since $\sec^2\beta = 1 + \tan^2\beta = 1 + \frac{1}{16} = \frac{17}{16}$, then $\cos^2\beta = \frac{16}{17}$, or $\cos\beta = \frac{4}{\sqrt{17}}$ and $\sin\beta = -\sqrt{1-cos^2\beta} =-\sqrt{1-\frac{16}{17}} =-\frac{1}{\sqrt{17}}$, since the angle β is in the fourth quadrant. Finally, $\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta = \frac{2}{3}\left(\frac{4}{\sqrt{17}}\right) - \frac{\sqrt{5}}{3}\left(-\frac{1}{\sqrt{17}}\right) = \frac{8+\sqrt{5}}{3\sqrt{17}} = \frac{8+\sqrt{5}}{3\sqrt{17}} \cdot \frac{\sqrt{17}}{\sqrt{17}}$
$= \frac{8\sqrt{17}+\sqrt{5}\sqrt{17}}{51}$

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16. What is the remainder when 2^{1000} is divided by 13?

(a) 2
(b) 3
(c) 4
(d) 10
(e) 12

Correct Answer: (b) 3
Solution:
When we divide 2^0 by 13, remainder is 1
When we divide 2^1 by 13, remainder is 2
When we divide 2^2 by 13, remainder is 4
When we divide 2^3 by 13, remainder is 8
When we divide 2^4 by 13, remainder is 3
When we divide 2^5 by 13, remainder is 6
When we divide 2^6 by 13, remainder is 12
When we divide 2^7 by 13, remainder is 11
When we divide 2^8 by 13, remainder is 9
When we divide 2^9 by 13, remainder is 5
When we divide 2^{10} by 13, remainder is 10
When we divide 2^{11} by 13, remainder is 7
When we divide 2^{12} by 13, remainder is 1

Now the cycle of remainders has started to repeat again i.e. 1, 2, 4, 8, 3, 6, 12 … until we have 1 again. We also know that every power that is a multiple of 12 will have a remainder of 1. We also know that if the power is $12n + m$ (where n is a whole number and m is a whole number less than 12) the remainder will correlate with the remainder for 2^m. Since, $1000 = 12 \times 83 + 4$ and is of the form $12n + 4$, we know that the remainder will be the same as it was for 2^4. Thus the remainder is 3.

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17. An integer when divided by 2 leaves a remainder of 1, when divided by 3 leaves a remainder of 2, when divided by 4 leaves a remainder of 3, and when divided by 5 leaves a remainder of 4. Find the smallest such integer.

(a) 29
(b) 39
(c) 59
(d) 119
(e) None of the above

Correct Answer: (c) 59
Solution: The integer is one less than a multiple of 2, 3, 4, and 5, so $2 \cdot 3 \cdot 2 \cdot 5 = 60 - 1 = 59$.

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18. The following trigonometric function models the position of a rung on a waterwheel:

$$y = -20 \cdot \sin\left(\frac{\pi}{6}t\right) + 16$$

where t is in seconds and y is the number of feet above water level. What is the maximum height of any given rung?

(a) 12 feet
(b) 16 feet
(c) 20 feet
(d) 36 feet
(e) 40 feet

Correct Answer: (d) 36 feet
Solution: First we must identify the measurements according to the equation.
Amplitude: 20 feet (distance from middle to peak)
Vertical shift: up 16 feet (position of the sine wave center)
Horizontal shift: None
Period: $\frac{2\pi}{\frac{\pi}{6}} = 12$ seconds

Since the vertical shift is up 16 feet, the new "wave center" is $y = 16$. Therefore, the top of the waterwheel (maximum height) is $16 + 20 = 36$ feet above the water level (and, the bottom is -4 feet or 4 feet underwater).

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19. If one line is drawn in the (Euclidean) plane, it divides the plane into two regions. If two lines are drawn in the plane, they divide the plane into either three regions or four regions. What is the largest number of regions into which the plane can be divided if 10 lines are drawn?

(a) 10
(b) 11
(c) 20
(d) 36

(e) 56

Correct Answer: (e) 56
Solution: Three lines can divide the plane into up to seven regions. Given three such lines, imagine that a fourth line is drawn that intersects each of the other three lines. On each side of each line that the fourth one intersects, it takes a region and splits it into two regions. Thus this fourth line will have effectively created up to four new regions, giving a total of 7 + 4 = 11 regions. Similarly, a fifth line will create up to five new regions, for a total of 11+5 = 16 regions. Continued similar reasoning shows that 10 lines can create up to 56 regions.

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20. Suppose you are playing basketball outside and the ball bounces off of a nail in the road. The ball begins leaking air and you notice that the diameter of the ball is shrinking at a constant rate of 4 centimeters per minute. If the diameter of the ball was 24 centimeters when you began playing, how fast is the volume of the ball shrinking 2 minutes after hitting the nail?

(a) -800π cm^3/min
(b) -24π cm^3/min
(c) -256π cm^3/min
(d) -512π cm^3/min
(e) -1024π cm^3/min

Correct Answer: (d) -512π cm^3/min
Solution: The diameter of the ball was 24 cm before it started leaking. Two minutes later the diameter is 16 cm, thus r = 8 cm and $\frac{dr}{dt} = -2$ cm/min. We are asked to find $\frac{dV}{dt}$. Differentiating both sides of the formula for the volume of a sphere $\left(V = \frac{4}{3}\pi r^3\right)$ with respect to *t*, gives $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Substituting in the values for *r* and $\frac{dr}{dt}$ yields $V = -512\pi$ cm^3/min.

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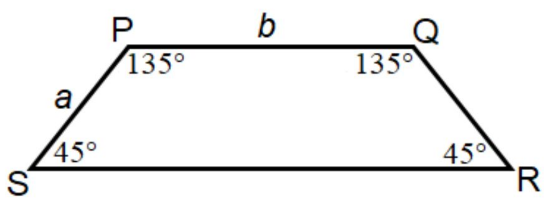
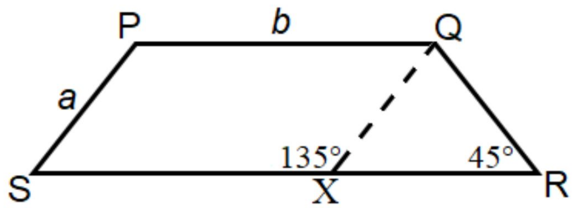
21. Trapezoid *PQRS* is shown below. How much greater is the area of this trapezoid than the area of a parallelogram with side lengths *a* and *b* and base angles of 45° and 135°?



(a) $\frac{1}{2}a^2$
(b) $\sqrt{2}a^2$
(c) $\frac{1}{2}ab$
(d) $\sqrt{2}ab$
(e) $2a$

Correct Answer: (a) $\frac{1}{2}a^2$
Solution: In the figure, draw a line segment from Q to the point X on $\overline{SR}$ of the trapezoid such that $\angle QXS$ has measure 135°, as shown in the figure below.



Since in trapezoid $PQRS$ $\overline{PQ}$ is parallel to $\overline{SR}$, it follows that $SPQX$ is a parallelogram with side lengths a and b and base angles of measure 45° and 135°. Thus, the area of the trapezoid is greater than a parallelogram with side lengths a and b and base angles of measure 45° and 135° by the area of $\triangle XQR$. Since $\angle QXS$ has measure 135°, it follows that $\angle QXR$ has measure 45°. Hence, $\triangle XQR$ is a 45°-45°-90° triangle with legs of length a. Therefore, its area is $\frac{1}{2}a^2$.

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22. If three fair dice are rolled, what is the probability that the sum of the three dice is 6?

(a) $\frac{5}{108}$
(b) $\frac{1}{18}$
(c) $\frac{1}{24}$
(d) $\frac{10}{36}$
(e) $\frac{1}{27}$

Correct Answer: (a) $\frac{5}{108}$
Solution: There are 216 possible combinations for sums (6 total on each dice). Here are the possible ways: (1,1,4), (1,4,1), (1,2,3), (1,3,2), (2,1,3), (2,3,1) (2,2,2), (3,2,1), (3,1,2), (4,1,1). Total possible for sum of six is 10. Therefore probability is $\frac{10}{216} = \frac{5}{108}$.

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23. Let a and b be positive numbers not equal to 1 such that $ab = a^b$ and $a^{2b} = \frac{a}{b}$. What is the value of $8a + 3b$?

(a) 27
(b) 29
(c) 30
(d) 32
(e) None of the above

Correct Answer: (b) 29
Solution: $ab = a^b$ implies $a^2b^2 = a^{2b} = \frac{a}{b}$. Then, by multiply both sides by $\frac{b}{a^2}$, we have $b^3 = \frac{1}{a}$. Raising both sides to the b power, we get $b^{3b} = \left(\frac{1}{a}\right)^b = \frac{1}{a^b} = \frac{1}{ab}$. Therefore, $ab = \frac{1}{b^{3b}}$. Also, from $b^3 = \frac{1}{a}$, we know $b^2 = \frac{1}{ab}$. Hence, $\frac{1}{b^2} = \frac{1}{b^{3b}}$ which means $b^2 = b^{3b}$, so $2 = 3b$, which implies $b = \frac{2}{3}$. Then, $a = \frac{1}{b^3} = \frac{27}{8}$. So $8a + 3b = 27 + 2 = 29$.

---

24. Simplify the following expression:

$$(\sin\theta + \cos\theta)^2$$

(a) $1 + \sin 2\theta$

(b) $\cos 2\theta$
(c) $2 \sin \theta \cos \theta - 1$
(d) $1 + \cos 2\theta$
(e) None of the above

Correct Answer: (a) $1 + \sin 2\theta$
Solution: Using the distributive property, multiply:

$$(\sin \theta + \cos \theta)^2 = (\sin \theta + \cos \theta)(\sin \theta + \cos \theta) = \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta$$

Using the property $\sin^2 \theta + \cos^2 \theta = 1$, simplify this result:

$$\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta = \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta = 1 + 2 \sin \theta \cos \theta$$

Now use the property of $\sin(a + b) = \sin a \cos b + \sin b \cos a$:

$$\sin 2\theta = \sin (\theta + \theta) = \sin \theta \cos \theta + \sin \theta \cos \theta = 2 \sin \theta \cos \theta$$

Therefore, $1 + 2 \sin \theta \cos \theta = 1 + \sin 2\theta$.

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25. Suppose $3x - y = 12$. What is the value of $\frac{27^x}{3^y}$?

(a) 3^{12}
(b) 9^4
(c) 27^2
(d) 3^8
(e) The value cannot be determined from the information given.

Correct Answer: (a) 3^{12}
Solution: We can rewrite 27^x as $(3^3)^x$ which is equivalent to 3^{3x}. The quotient can be rewritten as 3^{3x-y} and since $3x - y = 12$, we get the quotient is equal to 3^{12}.

---

26. Suppose that the number m is the product of all the positive prime numbers less than 40. How many positive divisors does m have?

(a) 412
(b) 24
(c) 4,096
(d) 37
(e) 1,600

Correct Answer: (c) 4,096.
Solution: There are exactly 12 prime numbers under 40, and we have $m = 2 \times 3 \times 5 \times \cdots \times 31 \times 37$. Each divisor of m is a product of the form $2^a \times 3^b \times 5^c \times \cdots \times 31^k \times 37^l$, where $a, b, c, \ldots, k, l$ are either 0 or 1. There are two choices

(0 or 1) for each of the 12 powers, so the total number of positive divisors is $2^{12} = 4{,}096$.

---

27. What is the probability that a solution for $x^2 + x < 42$ is also a solution for $x^2 - 14x > 15$?

(a) $\frac{6}{13}$
(b) $\frac{7}{13}$
(c) $\frac{2}{13}$
(d) $\frac{1}{13}$
(e) None of the above

Correct Answer: (a) $\frac{6}{13}$
Solution: We begin by solving $x^2 + x < 42$ and $x^2 - 14x > 15$:

$x^2 + x < 42$
$x^2 + x - 42 < 0$
$(x + 7)(x - 6) < 0$
$-7 < x < 6$

$x^2 - 14x > 15$
$x^2 - 14x - 15 > 0$
$(x + 1)(x - 15) > 0$
$x < -1$ or $x > 15$

The interval where the inequalities share possible solutions is $(-7, -1)$ and the interval containing the solutions to the first equation is $(-7, 6)$. So, the probability is the length of the interval from $(-7, -1)$ divided by the length of the interval from $(-7, 6)$. Thus, $\frac{6}{13}$ is the correct answer.

---

28. A **palindrome** is a positive integer which reads the same forwards as backwards. For example, 131 or 54845. What integer greater than 1 is a factor of every four-digit palindrome?

(a) 6
(b) 11
(c) 111
(d) 121
(e) None of the above

Correct Answer: (b) 11
Solution: $abba = 1000a + 100b + 10b + a$
$= 1001a + 110b$
$= 11(91a + 10b)$
Thus, all four-digit palindromes are divisible by 11.

---

29. The number of cubic units in the volume of certain cube is half as many as the number of square units in its surface area. Find the volume of the cube (in cubic units).

(a) 9
(b) 27
(c) 36
(d) 108
(e) 206

Correct Answer: (b) 27
Solution: If x is the side length of the cube, then the number of square units in its surface area is given by $6x^2$ and the number of cubic units is given by x^3. We are told that x^3 is half as many as $6x^2$, so $x^3 = 3x^2$. We can solve for x:

$x^3 = 3x^2$
$0 = 3x^2 - x^3$
$0 = (3 - x)x^2$

So $x = 0$ or $x = 3$, but since the cube can't have a side length of 0, it must be the side length is 3. So the volume is 27.

---

30. A snack pack of Starbursts contains two candies. The candies might be pink, yellow, orange, or red. Each color is equally likely to appear. What is the probability of opening a snack pack with at least one yellow candy?

(a) $\frac{1}{4}$
(b) $\frac{3}{16}$
(c) $\frac{3}{8}$
(d) $\frac{4}{10}$
(e) $\frac{7}{16}$

Correct Answer: (e) $\frac{7}{16}$
Solution: At least one yellow includes the cases of two yellows, one yellow/one not yellow, and one not yellow/one yellow. The probabilities of these cases are $\frac{1}{16} + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{7}{16}$.

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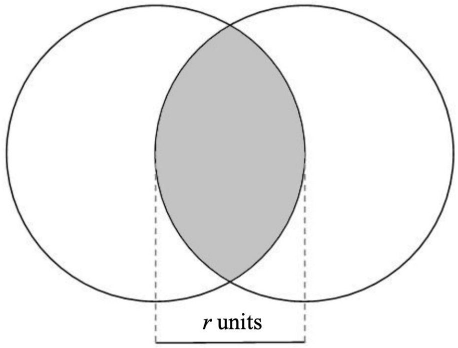
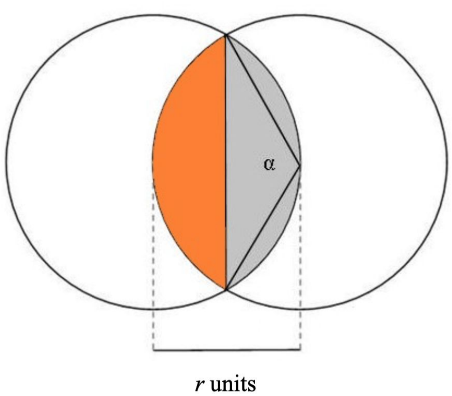
31. Two circles of radius r overlap such that the outer edge of each circle passes through the center of the other circle. What is the area of their overlap?



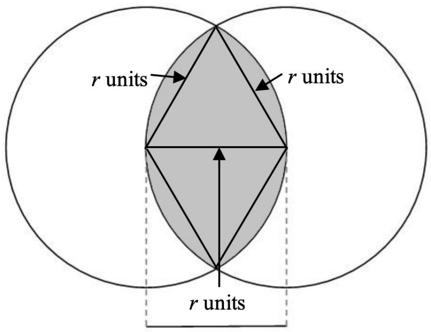
(a) $A = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) r^2$
(b) $A = \left(\frac{3\pi}{2} - \frac{\sqrt{3}}{2}\right) r^2$
(c) $A = \left(\frac{2\pi}{3} - \frac{1}{2}\right) r^2$
(d) $A = \left(\frac{3\pi - 1}{2}\right) r^2$
(e) None of the above

Correct Answer: (a) $A = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) r^2$

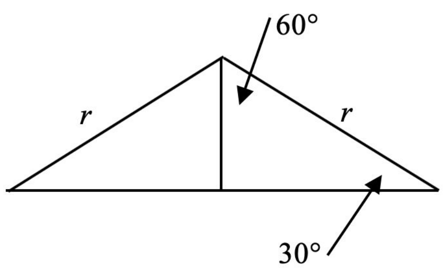
Solution: To find the area of the overlap, we will find the area of the sector of the circle subtended by angle α and subtract the area of the triangle. This will allow us to calculate the area shaded in orange in the figure at right (this is called a segment of a circle). Then, the total overlap of the two circles is two times this area.



First, find the measure of angle α. We know the two sides of the triangle which form angle α are both r units since they are radii of the circle. We can draw in the line segment connecting the centers of the two circles. This forms two equilateral triangles with all sides measuring r units. Since the triangles are equilateral, the angles are also equal. So, the angle α above is composed of two 60° angles or α is 120°.



Now, calculate the area of the sector of the circle subtended by a 120° angle. If we convert the angle measurement to radians, we can use the formula $A = \frac{1}{2}r^2\theta$. Then we still need to figure out the area of the triangle with an angle of 120° included between two sides measuring one unit.



We now need to find the area of the above triangle.

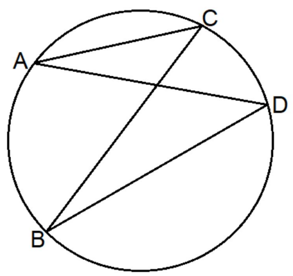
Angle α has been bisected by the line connecting the two centers of the circles, giving us two right triangles with 30° and 60° angles.

The height of the right triangle is $\frac{1}{2}r$ and the third side of it is $\frac{\sqrt{3}}{2}r$ which gives a base of $\sqrt{3}r$ for the triangle with α.

So the area of the segment of the circle is the area of the sector minus the area of the triangle. And the area of the overlap of the two circles is twice the area of one segment.

$$A = 2 \cdot \left(\frac{1}{2} \cdot r^2 \cdot \frac{120\pi}{180} - \frac{1}{2} \cdot \sqrt{3}r \cdot \frac{1}{2}r\right)$$
$$A = 2 \cdot \left(\frac{120\pi}{360}r^2 - \frac{\sqrt{3}}{4}r^2\right)$$
$$A = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2$$

32.
In the figure below, $\angle ACB$ and $\angle ADB$ are inscribed in the circle. Which of the following statements is true?



(a) The measure of $\angle ACB$ is greater than the measure of $\angle ADB$.
(b) The measure of $\angle ACB$ is less than the measure of $\angle ADB$.
(c) The measure of $\angle ACB$ is equal to the measure of $\angle ADB$.
(d) There is no relationship between the measure of $\angle ACB$ and the measure of $\angle ADB$.
(e) There is not enough information to determine the relationship between the measure of $\angle ACB$ and the measure of $\angle ADB$.

Correct Answer: (c) The measure of $\angle ACB$ is equal to the measure of $\angle ADB$.
Solution: Let the measure of arc AB be $x°$. Since $\angle ACB$ is inscribed in the circle and intercepts arc AB, the measure of $\angle ACB$ is equal to half the measure of arc AB. Thus, the measure of $\angle ACB$ is $\frac{x°}{2}$. Similarly, since $\angle ADB$ is also inscribed in the circle and intercepts arc, the measure of $\angle AYB$ is also $\frac{x°}{2}$. Therefore, the measure of $\angle ACB$ is equal to the measure of $\angle ADB$.

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33. Consider the equation $ax^2 + bx + c = 0$, where a, b, and c are constant with $a < 0$. Find the conditions on b and c such that the equation *always* has two real roots with opposite signs.

(a) $b = 0, c > 0$
(b) $b = 0, c = 0$
(c) $b = 0, c < 0$
(d) $b > 0, c > 0$
(e) Both (a) and (d)

Correct Answer: (e) Both (a) and (d)
Solution: Case for answer choice (a): We can use the expression $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the roots of the equation. The key to having two real roots with opposite signs is to have the determinant ($b^2 - 4ac$) be positive. Otherwise you will be taking the square root of a negative number and the roots would not be real numbers. In this case, $b = 0$, so we must determine if $-4ac$ is positive, so we can have real roots. Since a is negative and c is positive, we have that $-4ac$ is positive and the determinant is positive. Thus, the numerators of the roots have opposite signs. Therefore, the roots have opposite signs.

Case for answer choice (d): Similar to the case for answer choice (a), we need the determinant ($b^2 - 4ac$) be positive. Thus, b^2 is positive in this case and so we again need $-4ac$ to be positive. (If $-4ac$ was negative, there is a possibility that $b^2 - 4ac$ can still be positive if $-4ac$ is smaller than b^2, but this is not always the case.) Because a is negative and c is positive, $-4ac$ will be positive and the determinant will be positive. Thus, the numerators of the roots have opposite signs. Therefore, the roots have opposite signs.

---

34. How many real roots does the equation $9^{2x^2} - 9^{x^2} - 2 = 0$ have?

(a) Two
(b) Three
(c) None
(d) Four
(e) None of above

Correct Answer: (a) Two
Solution: $9^{2x^2} - 9^{x^2} - 2 = 0$
Factor: $\left(9^{x^2} + 1\right)\left(9^{x^2} - 2\right) = 0$
Solving $9^{x^2} + 1$: No solution for $x \in \mathbf{R}$
Solving $9^{x^2} - 2$ you have two real roots: $x = \sqrt{\frac{ln\,2}{2\,ln\,3}} \approx 0.56166...$, $x \approx -\sqrt{\frac{ln\,2}{2\,ln\,3}} \approx -0.56166...$

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35. Given: Set $A = \left\{ \textit{All values of x such that } \frac{x^2-9}{x+2} \le 0 \right\}$
Set $B = \{ \textit{All values of x such that } 3x^2(x+3)(x^2-6x+9) \le 0 \}$
Set $C = \left\{ \textit{All values of x such that } \frac{(x-3)(x-2)^3}{x+3} \le 0 \right\}$
Find: $A \cap B \cap C$

(a) $(-\infty, -3]$
(b) $(-\infty, -3)$
(c) $(-\infty, -3) \cup [2, 3]$
(d) $(-\infty, -3] \cup (-2, 3]$
(e) None of the above

Correct Answer: (e) None of the above
Solution: Set $A = \left\{ \textit{All values of x such that } \frac{x^2-9}{x+2} \le 0 \right\} = (-\infty, -3] \cup (-2, 3]$
Set $B = \{ \textit{All values of x such that } 3x^2(x+3)(x^2-6x+9) \le 0 \} = (-\infty, -3] \cup 0 \cup 3$
Set $C = \left\{ \textit{All values of x such that } \frac{(x-3)(x-2)^3}{x+3} \le 0 \right\} = (-\infty, -3) \cup [2, 3]$
Thus, $A \cap B \cap C = (-\infty, -3) \cup 3$.

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36. How many solutions does $\sin x = 0.01x^2$ have?

(a) 3
(b) 4
(c) 5
(d) 6

(e) 7

Correct Answer: (d) 6.
Solution: We are looking for where the two functions intersect. Since the range of sin x is between –1 and 1, you only need to look where $0.01x^2$ is less than 1. This would be where x is between –10 and 10. On the positive x-axis sine is negative at this point (which is just past 3π) so there would be 4 intersections since there are two positive cycles of sine, each of which have 2 intersections. On the negative x-axis there is only one positive cycle so this would add 2 more intersections for a total of 6.

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37. A radiator has a capacity of C liters and is full with L percent antifreeze solution. How much of this should be drained and then replaced with H percent antifreeze solution to make a T percent
antifreeze solution. (Note: $L < T < H$.)

(a) $V = \frac{C(T-L)}{(H-L)}$
(b) $V = \frac{C(H-L)}{(T-L)}$
(c) $V = \frac{C(L-T)}{(L-H)}$
(d) $V = \frac{C(L-H)}{(L-T)}$
(e) None of the above

Correct Answer: (a) $V = \frac{C(T-L)}{(H-L)}$
Solution: $LC - LV + HV = CT$

$$-LV + HV = CT - LC$$
$$V(H-L) = C(T-L)$$
$$V = \frac{C(T-L)}{(H-L)}$$

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38. Find GCF(a, b) [that is, *the greatest common factor*], given:

a = 1, 111, …, 111 (100 ones in a row total) and
b = 11,111,111.

(a) 1
(b) 11
(c) 101
(d) 1,111
(e) 10,001

Correct Answer: (d) 1,111
Solution: We can apply the Euclidean algorithm twice. First we see that:
1,111,...,111 = 1,111,...,110,000 + 1,111, where the first number contains 96 ones in a row followed by 4 zeros. Then, we factor out 11,111,111 from our 96 ones in a row and 4 zeros to get $11{,}111{,}111(10^{84} + 10^{72} + 10^{64} + \cdots + 10^{10} + 10^{4}) + 1{,}111$. This means the

GCF(a, b) = GCF(11,111,111; 1,111). And 11,111,111 = 1,111(10^4 + 1) + 0,
which means 1,111 evenly divides 11,111,111 and is GCF(11,111,111; 1,111), and hence is GCF(a, b).

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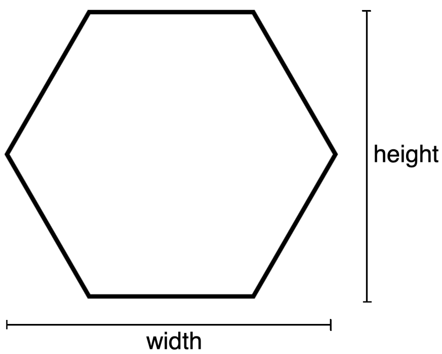
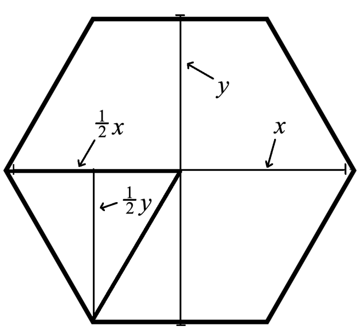
39. In a regular hexagon, oriented like the one below, the height is increased by $66\frac{2}{3}$ % and the width is decreased by 60%. What is the percent decrease in the area of the resulting hexagon?



(a) $33\frac{1}{3}$ %
(b) 25%
(c) 40%
(d) 50%
(e) 75%

Correct Answer: (a) $33\frac{1}{3}$ %
Solution:



The area of the regular hexagon can be found by finding the area of one equilateral triangle and then multiplying it by six.

Area of 1 equilateral triangle: $A = \frac{1}{2}bh = \frac{1}{2}\left(\frac{1}{2}x\right)\left(\frac{1}{2}y\right) = \frac{1}{8}xy$. Area of 6 equilateral triangles (and thus the original regular hexagon): $A = 6 \cdot \frac{1}{8}xy = \frac{6}{8}xy = \frac{3}{4}xy$.

The transformed hexagon has the height increase by $66\frac{2}{3}$ % or $1.\overline{6}y$ and the width decrease by 60% or $0.4x$. Thus, the area of the transformed hexagon is $A = \frac{3}{4}\left(\frac{5}{3}y\right)\left(\frac{2}{5}x\right) = \frac{2}{4}xy$. Therefore, the percent decrease from $\frac{3}{4}$ to $\frac{2}{4}$ is $33\frac{1}{3}$ %.

40. A number N is a product of 10 different prime numbers ($N = p_1 \cdot p_2 \cdot \ldots \cdot p_{10}$). If M is the number of divisors strictly larger than 1, then:

(a) M is a perfect square
(b) M is odd
(c) M is even
(d) $M < 1000$
(e) None of the above

Correct Answer: (b) M is odd.
Solution: Some people might be able to prove that there will be $2^{10} - 1$ divisors of N which are different from 1. Could this be too much to ask? Actually, we're asking for less and there's a straightforward way to do it. There are as many divisors of N consisting of one prime factor as divisors with 9 prime factors. So together we'll form an even number of factors. Same for the divisors with 2 prime factors and 8 prime factors. Then again same number of divisors with 3 factors as divisors with 7 prime factors. Then 4 prime factors and 6 prime factors in the same amount. We can see there are C(10, 5) divisors with 5 prime factors which is an even number again. So, so far we get even number of divisors, except that we did not include the full number N. Therefore, we have an odd number of divisors.