State Math Contest 2018 Senior Exam

Weber State University

March 8, 2018

Instructions:

- Do not turn this page until your proctor tells you.
- Enter your name, grade, and school information following the instructions given by your proctor.
- Calculators are not allowed on this exam.
- This is a multiple choice test with 40 questions. Each question is followed by answers marked a), b), c), d), and e). Only one answer is correct.
- Mark your answer to each problem on the bubble sheet answer form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.
- Scoring: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- You will have 2 hours and 30 minutes to finish the test.
- You may not leave the room until at least 10:15 a.m.

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1. The front chain wheel of a bicycle has radius eight inches, and the rear chain wheel has radius four inches. The radius of the bicycle's wheels is 32 inches. If a cyclist pedals at the rate of three revolutions per second, how fast is she going?

a) 8π feet/second b) $128\pi^2$ feet/second c) 32π feet/second

d) 64 feet/second e) 96 feet/second

2. A cold-water faucet can fill a sink in 12 minutes, and a hot-water faucet can fill the same sink in 15 minutes. The drain at the bottom of the sink can empty the sink in 25 minutes. If both faucets and the drain are open, how long will it take to fill the sink?

a) $5\frac{15}{57}$ minutes b) $7\frac{3}{4}$ minutes c) $9\frac{1}{11}$ minutes

d) $10\frac{4}{11}$ minutes e) $20\frac{4}{57}$ minutes

3. Three friends go to a restaurant for lunch. At the end of the meal, each of the three friends flips a fair coin, getting either a head or a tail. If two of the three coin flip outcomes match, the person with the non-matching outcome pays for everyone's lunch. If all three coin flip outcomes are the same, each friend flips a fair coin again. What is the probability that each friend has to flip a coin more than two times before it is decided which friend pays for lunch?

a) $\frac{1}{16}$ b) $\frac{1}{4}$ c) $\frac{1}{3}$

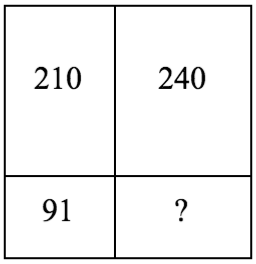
d) $\frac{9}{16}$ e) $\frac{3}{4}$

4. Ryan, Eric, and Kim like to hike mountains. Kim hiked six mountains that neither Ryan nor Eric hiked. There were only four mountains that all three hikers climbed. There was only one mountain Eric and Ryan hiked that Kim did not hike. There were no mountains that only Eric and Kim hiked. Ryan hiked six times more mountains than Eric. Kim hiked $\frac{1}{3}$ of the mountains that Ryan hiked and two times more mountains that Eric hiked. How many mountains did only Ryan hike?

a) 22 b) 27 c) 29

d) 48 e) 56

5. A rectangle is partitioned into 4 subrectangles as shown below. If the subrectangles have the indicated areas, find the area of the unknown rectangle.



a) 78 b) 98 c) 104

d) 270 e) 390

6. What is the coefficient of x^2y^4 when expanding the binomial $(2x+y)^6$?

a) 60 b) 30 c) 15

d) 120 e) 12

7. Memphis is due north of New Orleans. Find the surface distance between Memphis (35°north latitude) and New Orleans (30°north latitude). Assume that the radius of Earth is 3960 miles.

a) 110π miles b) 19,800 miles c) 55π miles

d) 7922 miles e) 55 miles

8. Two ships leave port at the same time, one heading due south and the other heading due east. Several hours later, they are 170 miles apart. If the ship traveling south traveled 70 miles farther than the other ship, how many miles did each ship travel?

a) Eastbound ship: 60 miles
Southbound ship: 130 miles

b) Eastbound ship: 65 miles
Southbound ship: 135 miles

c) Eastbound ship: 70 miles
Southbound ship: 140 miles

d) Eastbound ship: 75 miles
Southbound ship: 145 miles

e) Eastbound ship: 80 miles
Southbound ship: 150 miles

9. What is the difference between the sum of the first 500 positive even numbers and the first 500 positive odd numbers?

a) 5 b) 10 c) 100

d) 500 e) 1000

10. Jamie has 100 feet of fencing material to enclose a rectangular exercise run for her dog. One side of the run will border her house, so she will only need to fence off the other three sides. What dimensions will give the enclosure the maximum area?

a) 20 ft $\times$ 20 ft b) 20 ft $\times$ 50 ft c) 25 ft $\times$ 25 ft

d) 25 ft $\times$ 50 ft e) 50 ft $\times$ 50 ft

11. Eratosthenes of Cyrene, Greece (276−194 BCE) was a scholar who lived and worked in Cyrene and Alexandria, Egypt. One day while visiting Syene, Egypt, he noticed that at noon the Sun's rays shone directly down a well. On this date one year later, in Alexandria, at noon he measured the angle of the Sun to be about 7.2 degrees. He estimated the distance between Alexandria and Syene to be 500 miles. Use this information to approximate the circumference of Earth.

a) $7,920\pi$ miles b) 24,000 miles (c) 133.3π miles

d) 25,000 miles e) $\pi(3960)^2$ miles

12. Let $z = -2 + 2i$ where $i^2 = -1$. Determine the value of z^4.

a) -64 b) $-16 + 16i$ c) 64

d) $16 + 16i$ e) $16 - 16i$

13. How many solutions does the equation

$$\sin(x) + \sin(2x) = 0$$

have in the interval $[0, 2\pi]$?

a) 5 b) 4 c) 3

d) 2 e) 0

14. Convert 5432.1 in base 8 to base 10. What is the hundreds digit?

a) 2 b) 4 c) 0

d) 8 e) 1

15. It takes 867 digits to number the pages of a book. How many pages are there in the book?

a) 867 b) 434 c) 375

d) 325 e) 189

16. Find the next number in the sequence $\frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{10}{11}, \ldots$

a) $\frac{11}{12}$ b) $\frac{16}{17}$ c) $\frac{12}{15}$

d) $\frac{12}{13}$ e) None of the above are possible.

17. If the number 18! is divided by 19, what will be the remainder?

a) 1 b) 3 c) 6

d) 12 e) 18

18. There are six novels that you want to read over the summer. Two of the novels are by the same author and you don't want to read those directly after one another. In how many orders can you read the six novels?

a) 36 b) 240 c) 480

d) 600 e) 720

19. A fair coin is tossed 4 times. What is the probability that exactly one run of 2 straight heads occurs?

a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $\frac{1}{2}$

d) $\frac{5}{16}$ e) $\frac{7}{16}$

20. What is the area of a triangle with sides of length 5, 5, and 8?

a) 6 b) 12 c) 20

d) 25 e) 40

21. Find all the values of x which satisfy the inequality $\sqrt{(x-2)^2} < |x|$.

a) $(-2, 0)$ b) $(-2, 1) \cup (2, \infty)$ c) $(1, \infty)$

d) $(2, 5)$ e) None of the above are possible.

22. How many positive integers less than or equal to 60 have either 2, 3, or 5 as a factor?

a) 12 b) 30 c) 44

d) 62 e) 72

23. Solve the equation $\sqrt[3]{7^{x-1}} = \sqrt{7^{2x+1}}$.

a) $x = \frac{5}{4}$ b) $x = -\frac{5}{4}$ c) $x = \frac{4}{5}$

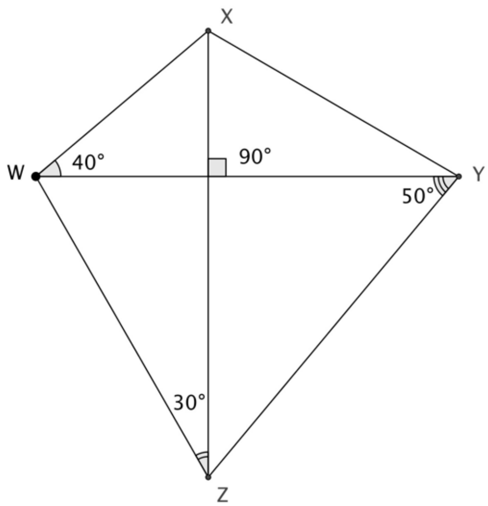
d) $x = -\frac{4}{5}$ e) None of the above are possible.

24. If the quadratic function $y = x^2 + ax + |a|$ has two distinct real zeros, then the interval for the value a is

a) $(-\infty, 0) \cup (4, \infty)$ b) $(-\infty, -4) \cup (4, \infty)$ c) $(-4, 4)$

d) $(-\infty, \infty)$ e) $(0, \infty)$

25. What is the measure of $\angle ZXY$ in the given figure?



a) Impossible to determine　b) $45°$　c) $50°$

d) $55°$　e) $60°$

26. Seven friends took a quiz. Each got a score that is a whole number between 1 and 100. No two friends got the same score. The median score received by the friends was 50 and the range (the maximum score minus the minimum score) was 20. What is the highest score that any of the friends could have received?

a) 50　b) 53　c) 60

d) 67　e) 99

27. Find the last digit of 3^{999}.

a) 1　b) 3　c) 7

d) 9　e) None of the above are possible.

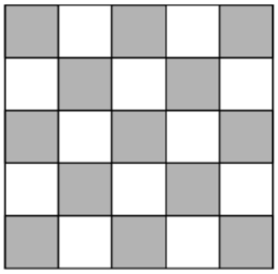
28. Let a, b, c, and d be positive integers and let $\log_a d = 24$. What is numerical value of

$$\log_c \tfrac{1}{d} \cdot \log_{a^{1.2}} \sqrt{b} \cdot \log_{b^{-2}} c^5?$$

a) $-\frac{3}{2}$　b) 1　c) $\frac{3}{2}$

d) 24　e) 25

29. How many rectangles can be made using the squares on a checkerboard that is 5 squares by 5 squares?



a) 64 b) 81 c) 256

d) 625 e) 225

30. Find the slope of the line through the point $P = (1, 1)$ on the parabola $y = x^2$ which does not intersect this parabola at any other point.

a) 0 b) 1 c) 2

d) -1 e) None of the above are possible.

31. Solve for x:

$$x - 4 - \frac{3x}{x+4} = \frac{-1}{1 - \frac{1}{1+\frac{1}{2}}}$$

a) -8 b) -2 and 2 c) -4 and 4

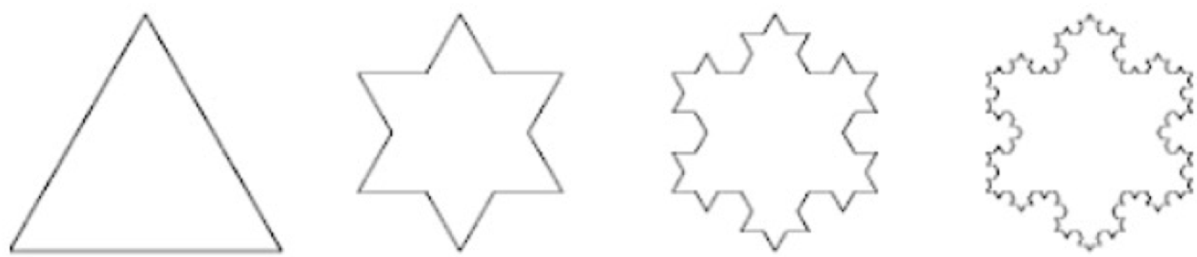
d) $\frac{3 \pm \sqrt{61}}{2}$ e) There are no real solutions.

32. Let n be a positive integer and define a function f such that $f(n) =$ the product of the digits of n. Which of the following cannot be a value of f?

a) 0 b) 1 c) 99

d) 100 e) 10^{100}

33. The image below shows the first four stages of creating a Koch Snowflake. It is formed by beginning with an equilateral triangle and then adding to each side of that triangle another triangle one ninth of the size of the previously added triangle. The table below shows how the area of the Koch Snowflake grows in a geometric sequence pattern. Note the table uses an initial side length of 1 unit.



Stage	Area
1	$\frac{\sqrt{3}}{4}$
2	$\frac{\sqrt{3}}{4}\left(1+3\cdot\frac{1}{9}\right)$
3	$\frac{\sqrt{3}}{4}\left(1+\frac{3}{9}+\frac{3\cdot 4}{9^2}\right)$
4	$\frac{\sqrt{3}}{4}\left(1+\frac{3}{9}+\frac{3\cdot 4}{9^2}+\frac{3\cdot 4^2}{9^3}\right)$
n	$\frac{\sqrt{3}}{4}\left(1+\sum_{n=1}^{n}\frac{3}{9}\cdot\frac{4^{n-1}}{9^{n-1}}\right)$

What is the limit of the area of the Koch Snowflake shown in the table?

a) $\frac{\sqrt{3}}{4}$ b) $\frac{3\sqrt{3}+4}{12}$ c) $\frac{2\sqrt{3}}{5}$

d) $\frac{\sqrt{3}}{3}$ e) ∞

34. The equation for a particular parabola is given by $y = \frac{x^2-6x+9}{-12} + 1$. A certain circle has the following properties: the center of the circle is at the focus of that parabola, and the circle intersects that parabola's vertex. Find the equation of the circle.

a) $(x-2)^2 + (y-3)^2 = 9$ b) $(x-2)^2 + (y+3)^2 = 3$

c) $(x-3)^2 + (y+2)^2 = 3$ d) $(x-3)^2 + (y+2)^2 = 9$

e) $(x-3)^2 + (y-2)^2 = 9$

35. Consider a polynomial $p(x)$ with integer coefficients. Let us assume that there are three distinct integers a_1, a_2, and a_3 such that $p(a_1) = p(a_2) = p(a_3) = 1$. At most how many integer roots does $p(x)$ have?

a) 0 b) 1 c) 2

d) 3 e) 5

36. Consider the line of equation $y = a - x$ for $a > 0$. The only value for the constant $a > 0$ that will make the line tangent to the circle of equation $x^2 + y^2 = 1$ is:

a) 1 b) $\sqrt{2}$ c) 2

d) 3 e) None of the above are possible.

37. $(3\sqrt{7} + 4)^3 + (3\sqrt{7} - 4)^3 =$

a) 1 b) $3\sqrt{7}$ c) $21\sqrt{7}$

d) $111\sqrt{7}$ e) $666\sqrt{7}$

38. A 300-room hotel is two thirds filled when the nightly room rate is \$90. For each \$5 increase in cost results in 10 fewer occupied rooms. Find the nightly rate that will maximize income.

a) \$100 b) \$85 c) \$110

d) \$90 e) \$95

39. If $f(x-1) = (1-x)(x+2)(x-3)$, then one of the antiderivatives of $f(x+1)$ is

a) $\frac{x^4}{4} + \frac{4}{3}x^3 - \frac{x^2}{2} - 4x$ b) $-x4 - 4x^3 + x^2 + 4$

c) $-\frac{1}{8}(x^6 + 8x^5 + 14x^4 - 16x^3 - 31x^2 + 8x + 16)$ d) $\frac{1}{2}(x+1)^2$

e) None of the above are possible.

40. What is the minimum period of the function $f(x) = \frac{\cos(x)+1}{\cot(\frac{x}{2})} - \sin^2(x) \cdot \tan(\frac{x}{2})$?

a) $\frac{\pi}{2}$ b) π c) 2π d) 3π e) 4π