UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph. D. Preliminary Examination in Geometry / Topology
May 27, 2020.

Instructions. This exam has two parts, A and B, covering material from Math 6510 and 6520, respectively. To pass the exam, you will have to pass each part. To pass each part you will have to demonstrate mastery of the material. Not all questions are equally difficult.

Part A. Answer five of the following questions. Indicate which of the questions are to be graded. If you do more than five, only the first five will be graded. Each question is worth 20 points. A passing score is 60 on this part.

---

1. Let M be a smooth manifold and $\gamma : \mathbb{S}^1 \to M$ be a smooth embedding. Show that there exists a smooth vector field X on M that does not vanish on $\gamma(\mathbb{S}^1)$ and which is tangent to $\gamma(\mathbb{S}^1)$ at each point. (The last condition means that for each point $p = \gamma(t)$, the vector X_p is a multiple of $\gamma'(t)$.)

2. Let M^2 be a smooth two manifold without boundary and $i : M^2 \to \mathbb{R}^5$ be a smooth immersion. Show that there is a projection $\pi : \mathbb{R}^5 \to \mathbb{R}^4$ such that $\pi \circ i$ is an immersion.

3. Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the two sphere in three space. Consider the vector fields

$$V = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad W = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

   (a) Show V and W are tangent to $\mathbb{S}^2$, and hence by restriction define vector fields on $\mathbb{S}^2$.

   (b) Find the flow $\Theta_t^V(x, y, z)$ on $\mathbb{S}^2$ generated by V.

   (c) Compute $(\Theta_h^V)_* W$ on $\mathbb{S}^2$.

   (d) Compute the Lie Derivative $\pounds_V W$ on $\mathbb{S}^2$ in two ways.

4. Let M and N be a smooth orientable manifolds. Show that the maps $i_a : M \to \mathbb{R} \times M$ given by $i_a(x) = (a, x)$ and $\pi : \mathbb{R} \times M \to M$ given by $\pi(t, x) = x$ induce inverse maps in de Rham Cohomology. Conclude that if smooth maps $f : M \to N$ and $g : M \to N$ are smoothly homotopic, then they induce equal maps in de Rham Cohomology.

5. Suppose that M and N are smooth surfaces in $\mathbb{R}^3$ that intersect at a point p and that do not have the same tangent plane at that point. Show that p is not an isolated point of $M \cap N$.

6. Let α be a closed two form on $\mathbb{S}^4$. Prove that $\alpha \wedge \alpha$ vanishes at some point.

Part B. Answer as many questions as you can. For a pass you need to solve completely at least three.

7. State the homotopy lifting property of covering maps and use it to prove that the covering map $S^n \to \mathbb{R}P^n$ from the n-sphere to the real projective n-space is not nullhomotopic for any $n \geq 1$.

8. Let V be a continuous vector field on the unit ball $B^n \subset \mathbb{R}^n$ which is nowhere zero. Prove that there are points $x, y \in S^{n-1} = \partial B^n$ and positive numbers $a, b > 0$ such that $V(x) = ax$ and $V(y) = -by$ (so the vector field points outward at x and inward at y).

9. Let X be a connected CW complex and $A \subset X$ a connected subcomplex. Suppose that inclusion $A \subset X$ is π_1-injective (for some, hence all, basepoints in A). Let $p : \tilde{X} \to X$ be the universal cover of X. Prove that each component of $p^{-1}(A)$ is simply connected.

10. Let m, n be positive integers and view S^1 as the unit circle in the complex plane. Let X be the space obtained from the cylinder $S^1 \times [0, 1]$ by identifying points $(z, 0)$ and $(w, 0)$ if $z^n = w^n$, and identifying points $(z, 1)$ and $(w, 1)$ if $z^m = w^m$.

    (a) Use the Mayer-Vietoris sequence to compute $H_i(X)$ for all i.

    (b) Use the Seifert-van Kampen theorem to find a presentation for $\pi_1(X)$.

11. Suppose M is a closed connected orientable 5-manifold such that $H_1(M) = \mathbb{Z}/2\mathbb{Z}$ and $H_2(M) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Compute $H_i(M)$ and $H^i(M)$ for all i.

12. Below is a picture of a Δ-complex X obtained from the square subdivided with the diagonal by identifying all sides of the square with orientations as indicated. So X has one vertex, two edges and two 2-cells. Write down the simplicial cochain complex that computes $H^i(X; \mathbb{Z}/2\mathbb{Z})$. Compute the algebra $H^*(X; \mathbb{Z}/2\mathbb{Z})$ using the definition of the cup product. Note: X is homotopy equivalent to a familiar space but you are not allowed to go via that route.