University of Utah, Department of Mathematics
May 2020, Algebra Qualifying Exam

There are ten problems on the exam. You may attempt as many problems as you wish; five correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Show there is no simple group of order $448 = 2^6 \cdot 7$.

2. Suppose G is a finitely generated Abelian group such that the automorphism group of G is finite. Let T be the torsion subgroup of G. Describe all possibilities for G/T up to isomorphism.

3. Let $R = \mathbb{Z}[x]$ and suppose that $I = \langle x \rangle$ and that $J = \langle x, 5 \rangle$. How many elements are in $\mathrm{Tor}_i^R(R/I, R/J)$ for $i = 0, 1, 2$?

4. Suppose $R = \mathbb{F}_3[x]$. Let M be the module generated by three elements a, b, c subject to the three relations $-xa + x^2 b + (x^2 - 1)c = 0$, $xb + xc = 0$ and $xb + x^2 c = 0$. Find a direct sum of cyclic R-modules isomorphic to M.

5. Show that there are at most 5 groups of order 20.

6. Let K be an algebraically closed field. Let $f \in K[x, y]$ be irreducible and let $g \in K[x, y]$ be such that f does not divide g. Prove that there is a point $(k_1, k_2) \in K^2$ such that
$$f(k_1, k_2) = 0 \text{ and } g(k_1, k_2) \neq 0.$$

7. Find a polynomial $f(x) \in \mathbb{F}_2[x]$ such that $\mathbb{F}_2[x]/f(x)$ is a field with 8 elements.

8. Let R be the ring of continuous real-valued functions on the closed interval $[0, 1]$. What are the maximal ideals of R?

9. Let σ be the automorphism of $\mathbb{F}_5(t)$ sending t to $t + 1$. Let G be the subgroup of $\mathrm{Aut}(\mathbb{F}_5(t))$ generated by σ. Find an element $s \in \mathbb{F}_5(t)$ such that
$$\mathbb{F}_5(t)^G = \mathbb{F}_5(s).$$

10. Let $2^{1/4}$ be the unique positive real fourth root of of 2 in $\mathbb{R}$. Show that $\mathbb{Q}(2^{1/4})/\mathbb{Q}$ is not a Galois extension.