Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve completely at least three problems and score at least 30 points. For a pass you need to solve completely at least two problems and score at least 25 points. Carefully state any theorems you use.

1. Let \((B, \| \cdot \|)\) be a Banach space. Explain the inclusion of \(B\) into its double dual. Prove that it is an isometry onto the image.

2. Let \(X\) be a set and \(\mathcal{A}\) be an algebra on \(X\). Let \(\rho_0 : \mathcal{A} \to [0, \infty]\) be a pre-measure on \(\mathcal{A}\). Recall that \(\rho^*(S) = \inf \{ \sum_{i=1}^{\infty} \rho_0(A_i) : A_i \in \mathcal{A} \text{ for all } i \text{ and } S \subset \bigcup_{i=1}^{\infty} A_i \}\) is an outer measure. Define what it means that \(\rho_0\) is a pre-measure and show that if \(A \in \mathcal{A}\) then \(\rho^*(A) = \rho_0(A)\).

3. Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be two measure spaces and \(A\) be measurable with respect to \(\mathcal{M} \times \mathcal{N}\), the \(\sigma\)-algebra generated by measurable rectangles. Show that if \(A_1 \subset A_2 \subset \ldots\) satisfy that \((\mu \otimes \nu)(A_i) = \int_X \nu(\{y : (x, y) \in A_i\})d\mu(x)\) for all \(i\) then
\[(\mu \otimes \nu)(\bigcup_{i=1}^{\infty} A_i) = \int_X \nu(\{y : (x, y) \in \bigcup_{i=1}^{\infty} A_i\})d\mu(x).\]

4. Let \((X, \mathcal{M}, \mu)\) be a measure space. Give an example of \(f_1, \ldots : X \to \mathbb{C}\), measurable, which converge in measure to the zero function but do not converge \(\mu\text{-a.e.}\). Show that there exists a subsequence \(n_1, n_2, \ldots\) so that \(f_{n_1}, \ldots\) converges almost everywhere to the zero function.

5. In this problem, let \(\lambda\) denote Lebesgue measure on \(\mathbb{R}\) and \(\hat{\cdot}\) denote the Fourier transform (on \(\mathbb{R}\)). Show that if \(f, g\) are Lebesgue integrable and \(\hat{f} = \hat{g}\) then \(f = g\ \lambda\text{-a.e.}\).
6. Assume that $\mu : \mathcal{B}_{\mathbb{R}^4} \to [0, \infty]$ is a measure which is outer regular and $\sigma$-finite. Show that if $\mu$ is inner regular on open sets then it is inner regular on all Borel sets. Recall that a Borel measure on $\mathbb{R}^4$ is called outer regular if for any $A \in \mathcal{B}_{\mathbb{R}^4}$ we have

$$\mu(A) = \inf \{ \mu(U) : U \text{ is open and } A \subset U \}$$

and it is called inner regular if

$$\mu(A) = \sup \{ \mu(K) : K \text{ is compact and } K \subset A \}.$$